Computing the Betti Numbers of Arrangements

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Arrangements in Computational Geometry

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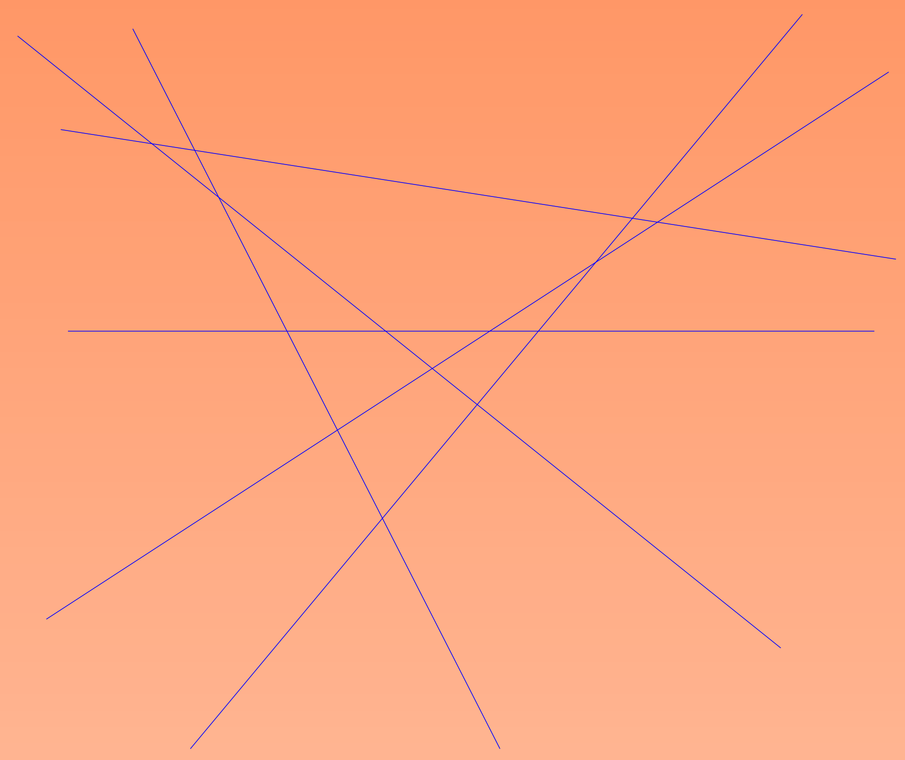
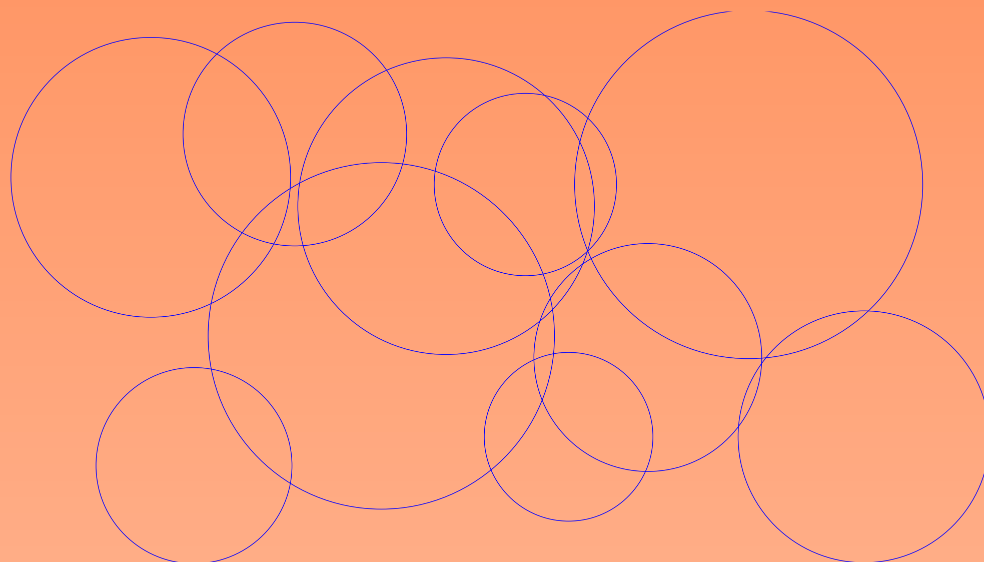
- Arrangements of semi-algebraic objects in $\mathbb{R}^k$, each defined by a fixed number of polynomials of constant degree.

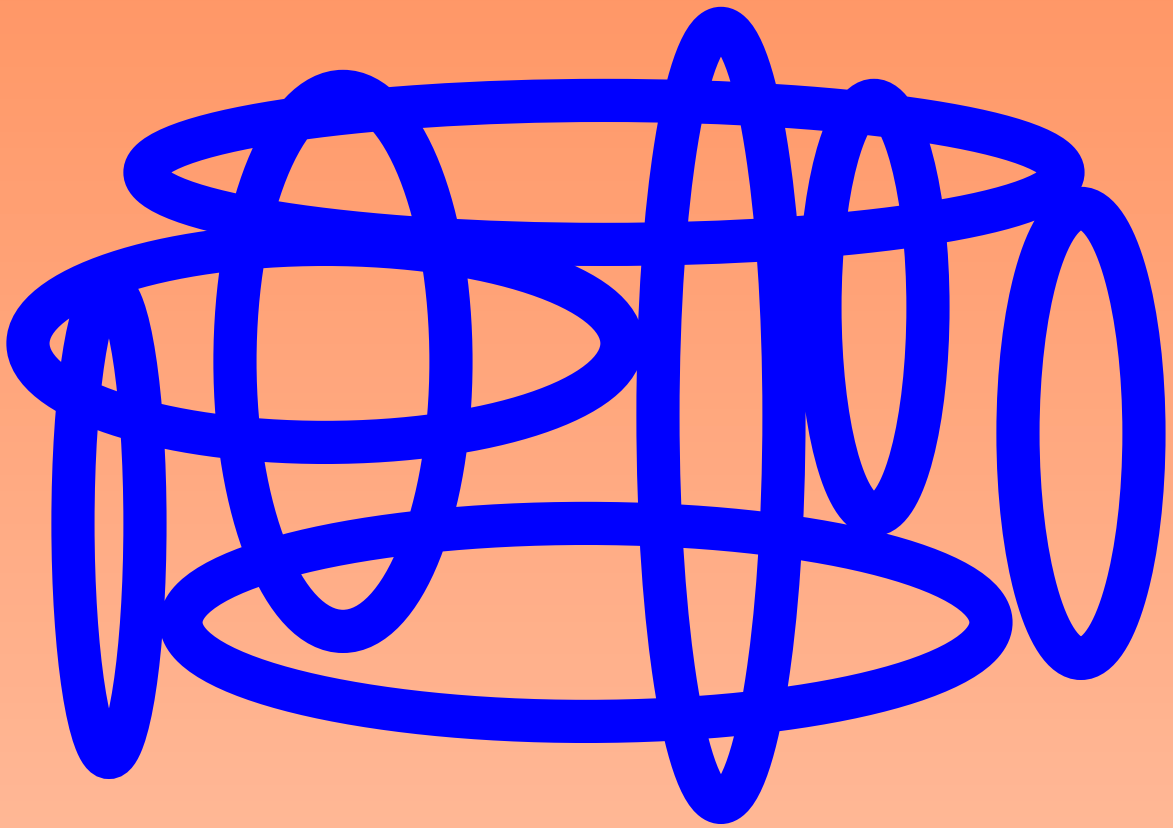
Arrangements of lines in the $\mathbb{R}^2$



Arrangement of circles in $\mathbb{R}^2$



Arrangement of tori in $\mathbb{R}^3$



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- $\beta_0(S) =$ the number of connected components.

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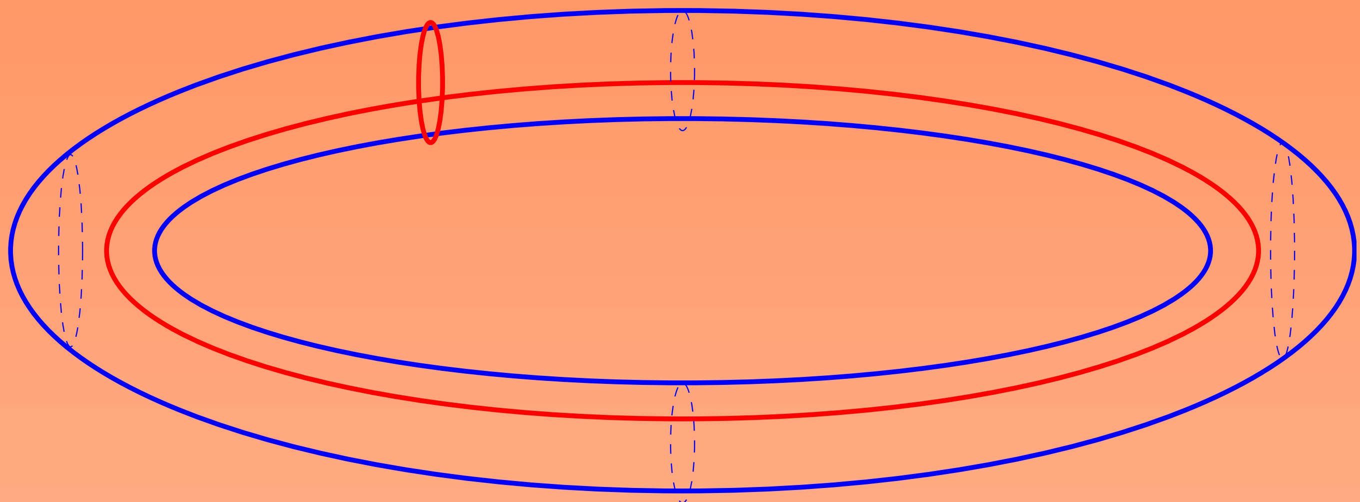

Betti Numbers of the Torus

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- $\beta_i(T) = 0, i > 2.$

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- Computing the Betti numbers of triangulated manifolds (Edelsbrunner, Dey, Guha et al).

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- We only count the number of algebraic operations and ignore the cost of doing linear algebra.

Two Approaches

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Global

vs

Local

First Approach (Global): Using Triangulations

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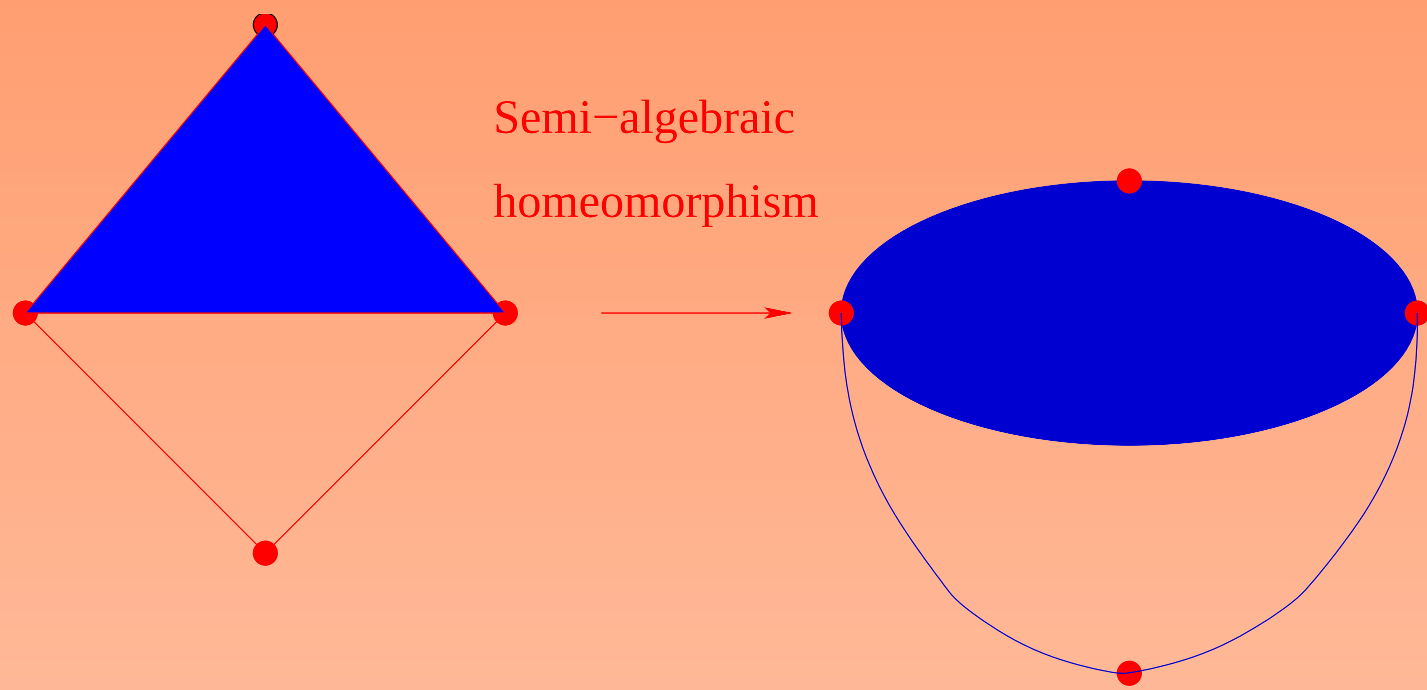
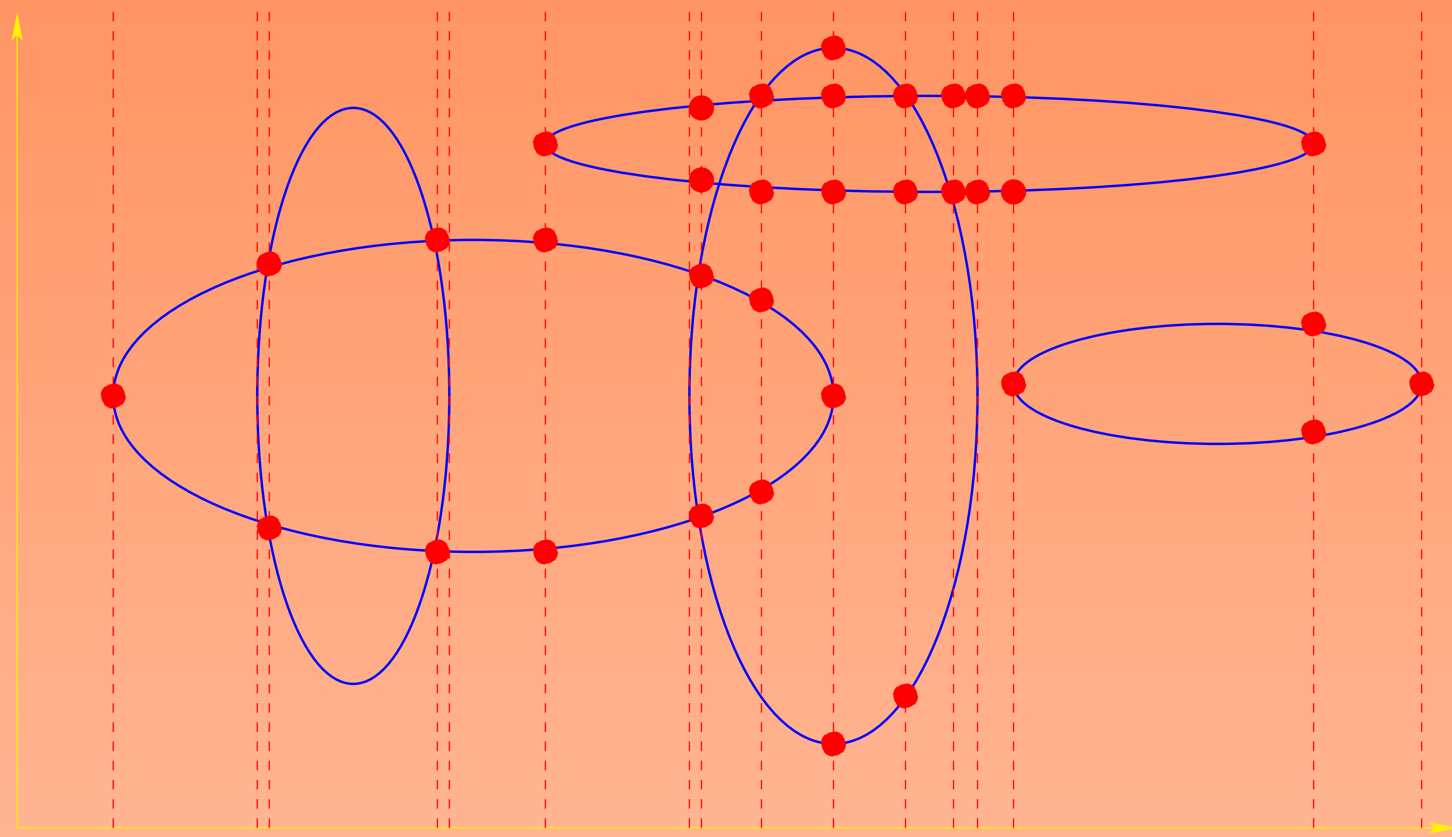


Triangulation via Cylindrical Algebraic Decomposition



Computing Betti Numbers using Global Triangulations

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- But ... CAD produces $O(n^{2^k})$ simplices in the worst case.

Second Approach (Local): Using the Nerve Complex

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- If the sets have the special property that all their non-empty intersections are contractible we can use the *nerve lemma* (Leray, Folkman).
- The homology groups of the union are then isomorphic to the homology groups of a combinatorially defined complex called the *nerve complex*.

The Nerve Complex

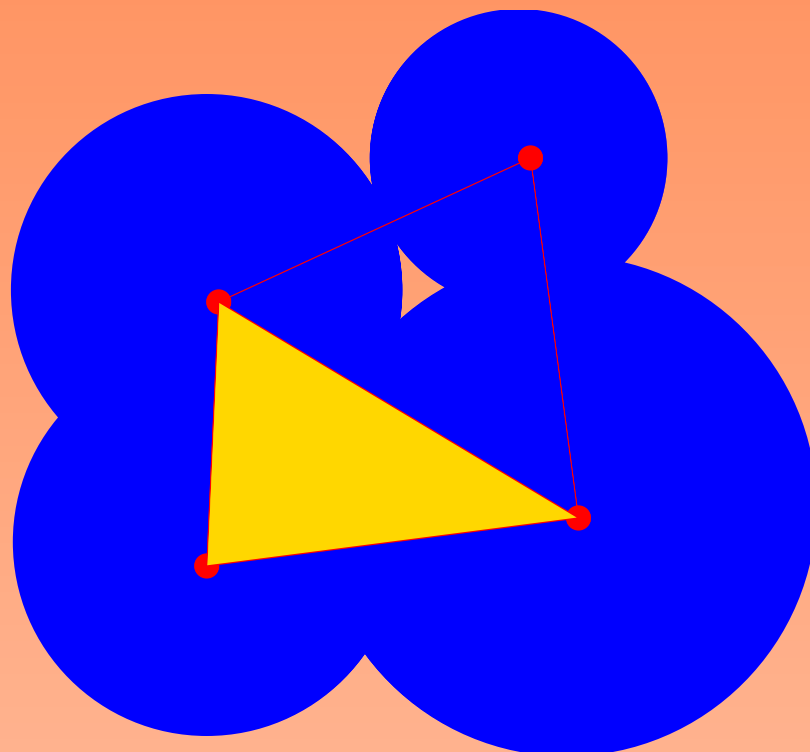

Figure 1: The nerve complex of a union of disks

Computing the Betti Numbers via the Nerve Complex (local algorithm)

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- Thus, the $(\ell+1)$-skeleton of the nerve complex can be computed by testing for non-emptiness of each of the possible $\sum_{1 \leq j \leq \ell+2} \binom{n}{j} = O(n^{\ell+2})$ at most $(\ell+2)$-ary intersections among the n given sets.

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- we can use the *Leray spectral sequence* as a substitute for the nerve lemma.
- This approach produced the first non-trivial bounds on the *individual* Betti numbers of arrangements rather than their sum (B, 2001).

Main Result

Theorem 1. *Let* $S_1, \ldots, S_n \subset \mathbb{R}^k$ *be compact semi-algebraic sets of constant description complexity and let* $S = \cup_{1 \leq i \leq n} S_i$, *and* $0 \leq \ell \leq k-1$. *Then, there is an algorithm to compute* $\beta_0(S), \ldots, \beta_\ell(S)$, *whose complexity is* $O(n^{\ell+2})$.

Complexes and Spectral Sequences

A crash course in homological algebra.

Double Complex

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
C^{0,2} & \xrightarrow{\delta} & C^{1,2} & \xrightarrow{\delta} & C^{2,2} & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
C^{0,1} & \xrightarrow{\delta} & C^{1,1} & \xrightarrow{\delta} & C^{2,1} & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
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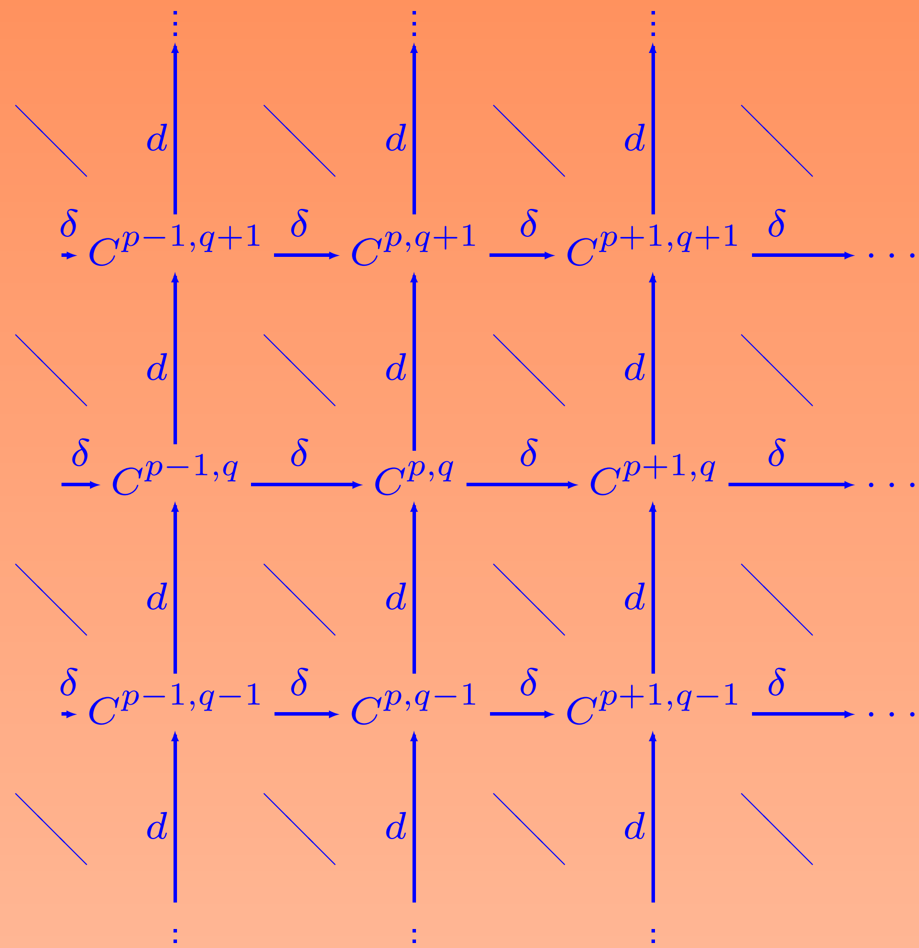
\end{array}
$$

The Associated Total Complex



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- $E_{r+1} = H(E_r, d_r)$,
- $E_\infty = H^*(\text{Associated Total Complex})$.

Spectral Sequence

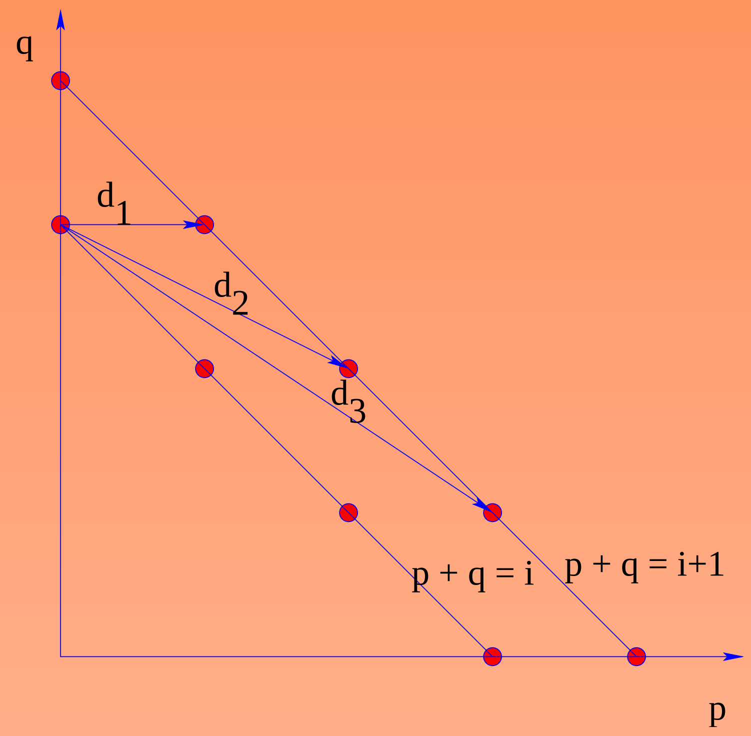

Figure 2: The differentials d_r in the spectral sequence (E_r, d_r)

The Mayer-Vietoris Double Complex I

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- Denote by $A_{\alpha_0,\ldots,\alpha_p}$ the sub-complex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.

The Mayer-Vietoris Double Complex II

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \prod_{\alpha_0} C^2(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^2(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \cdots \\
 & & \uparrow d & & \uparrow d & & \\
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 & & \uparrow d & & \uparrow d & & \\
 & & 0 & & 0 & &
\end{array}
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- In order to compute β_ℓ, we only need to compute upto $E_{\ell+2}$.*But the punchline is that:*
- In order to compute the differentials $d_r, 1 \leq r \leq \ell + 1$, it suffices to have *independent triangulations of the different unions taken $\ell + 2$ at a time.*

- For instance, it should be intuitively clear that in order to compute $\beta_0(\cup_i S_i)$ it suffices to triangulate pairs.

Open Problems

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- Other applications of spectral sequences, possibly in the theory of distributed computing ?