Algorithmic Semi-algebraic Geometry and its applications

Saugata Basu
School of Mathematics &
College of Computing
Georgia Institute of Technology.

Introduction: Three problems

1. Plan the motion of a robot with several degrees of freedom, amidst obstacles.

2. Find the possible geometric conformations of a molecule given the bond lengths and bond angles.

3. Given two ordered sets of n points in the plane, is it possible to change the first set continuously into the second maintaining the order type.

Semi-algebraic Sets

- Subsets of R^k defined by a formula involving a finite number of polynomial equalities and inequalities.

Semi-algebraic Sets

- Subsets of R^k defined by a formula involving a finite number of polynomial equalities and inequalities.

- A basic semi-algebraic set is one defined by a conjunction of weak inequalities of the form $P \geq 0$.

Semi-algebraic Sets

- Subsets of R^k defined by a formula involving a finite number of polynomial equalities and inequalities.
- A basic semi-algebraic set is one defined by a conjunction of weak inequalities of the form $P \geq 0$.
- They arise as configurations spaces (in robotic motion planning, molecular chemistry etc.), CAD models and many other applications in computational geometry.

Basic Properties of Semi-algebraic Sets

- Closed under union, intersection, complementation and projection.
- Most sets in R^k that arise in practice can be closely approximated by semi-algebraic sets (witness splines).
- Compact semi-algebraic sets are finitely triangulable.
- First order theory of the reals is decidable.

The Important Algorithmic Problems

Given a description of a semi-algebraic set $S \subset R^k$:

1. given two points $x, y \in S$, decide if they are in the same connected component of S and if so output a semi-algebraic path in S joining them,

2. compute semi-algebraic descriptions of the connected components of S,

3. compute topological invariants of S, e.g. its Euler characteristic, homology groups etc.

Outline of the talk

1. Algorithms.
   (a) Deciding connectivity questions.
2. Quantitative bounds on the complexity of semi-algebraic sets.
   (a) Bounds on Betti numbers.
   (b) Complexity of single cells and connections to computational geometry.

Complexity of Algorithms

The complexity of an algorithm is measured in terms of the following three parameters:

- the number of polynomials, n, used to define the input semi-algebraic set S,
- the maximum degree, d, of these polynomials and
- the number of variables, k.

Analogy with Semi-linear Geometry

- Consider the special case when all the input polynomials are linear and thus the given set is *semi-linear*.
- Algorithms for computing properties of semi-linear sets are widely studied in computational geometry.
- Typically, the complexities of these algorithms are of the order of $O(n^k)$ where n is the number of linear polynomials in the input.

- Motivates designing algorithms for semi-algebraic sets such that the *combinatorial complexity* (the part depending on n) matches that for the corresponding semi-linear problem.

- In the semi-algebraic case there is usually an additional algebraic overhead – *algebraic complexity* – of the order of $d^{O(k)}$ or $d^{O(k^2)}$.

Cylindrical Algebraic Decomposition

- Introduced by Collins (1976). Used by Schwartz and Sharir for solving the piano-mover's problem.

- Complexity is $(nd)^{2^{O(k)}}$ (doubly exponential) because of iterated projections.

Connectivity via Roadmaps

A roadmap of S, $R(S)$, is a semi-algebraic set of dimension at most one, satisfying

1. for every semi-algebraically connected component C of S, $C \cap R(S)$ is non-empty and semi-algebraically connected.

2. for every $x \in R$, and for every semi-algebraically connected component C' of S_x, $C' \cap R(S)$ is not empty.

Brief History

Grigor'ev-Vorobjov, Canny, Gournay-Risler, Heintz-Roy-Solerno.

B-Pollack-Roy, (1995) We give an algorithm to solve both problems for semi-algebraic set restricted to a variety of dimension k' in time,

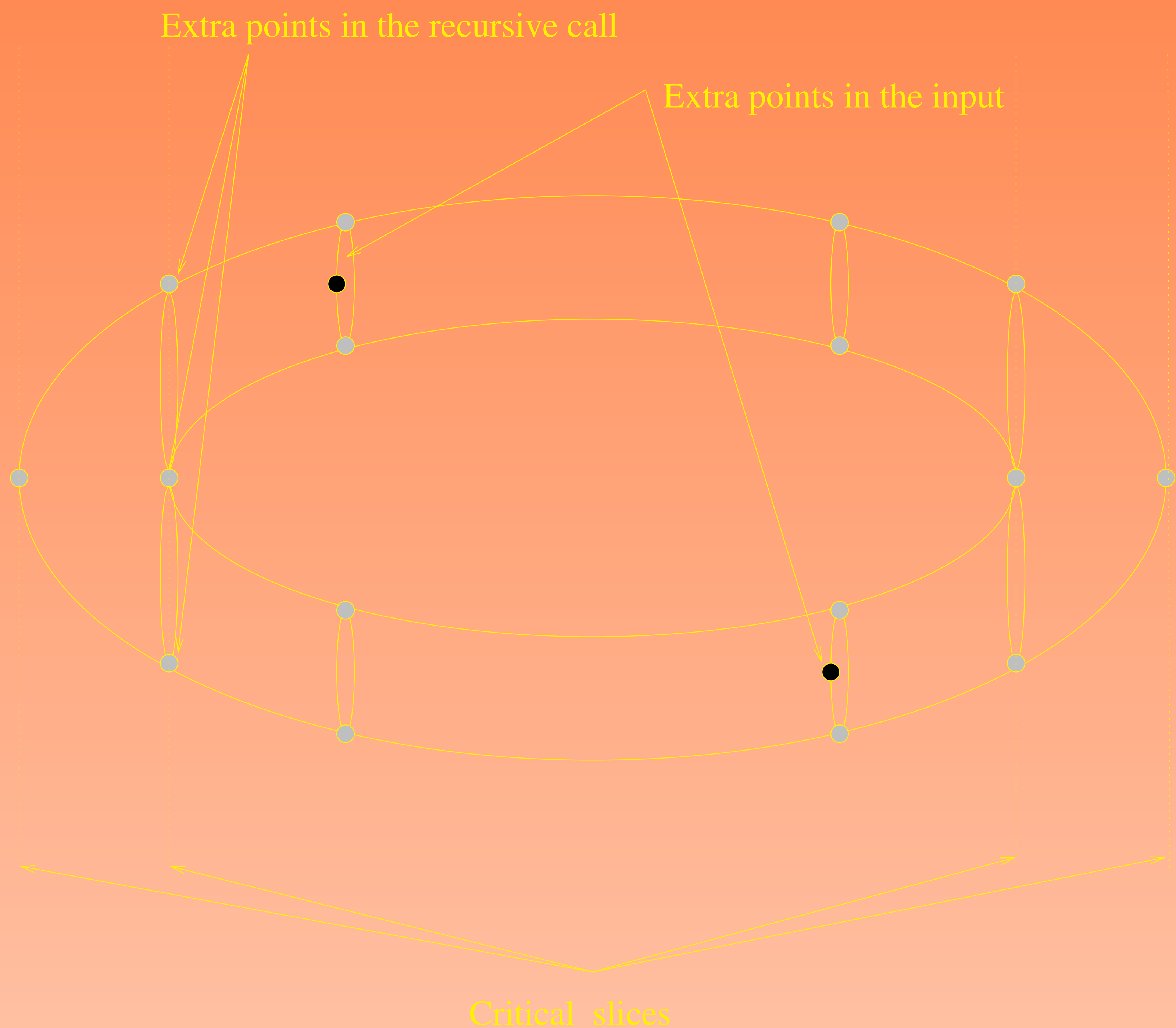
$$n^{k'+1}d^{O(k^2)}.$$

How to compute the roadmap ?

In case of a compact, smooth algebraic hypersurface S one can obtain the roadmap by:

1. Follow the X_2-extremal points in the X_1 direction.
2. Recurse at certain special slices corresponding to the critical values of the projection map onto the X_1 co-ordinate.

Example: Smooth torus in R^3


Extra points in the recursive call
Extra points in the input
Critical slices

Representing points

In our algorithms, whenever we compute a point

$$x = (x_1, \ldots, x_k)$$

what we actually compute is :

1. A univariate polynomial $f(t)$.

2. A root, say α, of f which is characterized by f and the sign vector

$$(\mathrm{sign}(f(\alpha)), \mathrm{sign}(f'(\alpha)), \ldots, \mathrm{sign}(f^{(deg(f)-1)}(\alpha))).$$

3. $k+1$ polynomials $g_0(t), \ldots, g_k(t)$, such that

$$x_i = \frac{g_i(\alpha)}{g_0(\alpha)}, 1 \leq i \leq k.$$

How to compute the roadmap ? (cont)

For a general algebraic set $Z(Q)$ one can obtain the roadmap by:

1. Parametrizing a procedure for computing a set of points guaranteed to meet every connected component of an algebraic set, treating X_1 as a parameter.

2. Recurse at certain special slices corresponding to the pseudo-critical values.

Pseudo critical values

How to compute the roadmap ? (cont)

For a general semi-algebraic set S we obtain the roadmap by:

1. Make perturbations such that no k of the input polynomials have a common real zero.
2. Computing roadmaps for all possible non-empty algebraic sets.
3. Recurse at certain special slices corresponding to the special values.

Connections with Computational Geometry: Arrangements

1. Arrangement of n lines in R^2.
   - Total combinatorial complexity : $O(n^2)$.
   - Combinatorial complexity of a single cell : $O(n)$.
2. Arrangement of n hyperplanes in R^k.
   - Total combinatorial complexity : $O(n^k)$.

- Combinatorial complexity of a single cell : $O(n^{\lfloor \frac{k}{2} \rfloor})$. (Consequence of the Upper Bound Theorem).

Arrangements of Surface Patches

- Each surface patch S_i is a *closed* semi-algebraic set contained in a hypersurface $Z(Q_i)$ and defined by a first-order quantifier-free formula involving a family of polynomials, $\{P_{i,1}, \ldots, P_{i,r}\}$.

- A *cell* is a maximal connected subset of the intersection of a fixed (possibly empty) subset of surface patches that avoids all other surface patches.

- The combinatorial complexity of an ℓ-dimensional cell C is the number of cells of dimension less than ℓ which are contained in the relative boundary of C.

Arrangement of circles in the plane

Known Results

1. For $k = 2$:
   - Complexity of the whole arrangement : $O(n^2)$.
   - Complexity of a single cell : $O(n\alpha(n))$. (Guibas, Sharir, Sifrony).
2. For $k = 3$:
   - Complexity of the whole arrangement : $O(n^3)$.

- Complexity of a single cell : $O(n^{2+\epsilon})$. (Halperin and Sharir).

3. Conjecture: Combinatorial complexity of a single cell is bounded by $O(n^{k-1}\alpha(n))$.

Topological Complexity of Semi-algebraic Sets

- An important measure of the topological complexity of a set S are the Betti numbers $\beta_i(S)$.
- Intuitively, $\beta_i(S)$ measures the number of i-dimensional holes in S.
- For example, if T is topologically a hollow torus, then $\beta_0(T) = 1, \beta_1(T) = 2, \beta_2(T) = 1, \beta_i(T) = 0, i > 2$,

Motivations

- As a measure "computational difficulty" of semi-algebraic sets. e.g. lower bounds for membership testing in terms of the sum of the Betti numbers (Yao et al.)

Motivations

- As a measure "computational difficulty" of semi-algebraic sets. e.g. lower bounds for membership testing in terms of the sum of the Betti numbers (Yao et al.)
- In studying the complexity of arrangements and their substructures in computational geometry.

Motivations

- As a measure "computational difficulty" of semi-algebraic sets. e.g. lower bounds for membership testing in terms of the sum of the Betti numbers (Yao et al.)

- In studying the complexity of arrangements and their substructures in computational geometry.

Topological Complexity of Semi-Algebraic Sets

Oleinik and Petrovsky (1949) Thom (1964) and Milnor (1965) proved that the sum of the Betti numbers of a semi-algebraic set $S \subset R^k$, defined by

$$P_1 \geq 0, \ldots, P_n \geq 0,$$

$$deg(P_i) \leq d, 1 \leq i \leq n,$$

is bounded by

$$(O(nd))^k.$$

This bound is tight as $\beta_0(S)$ could be as large.

What about a single connected component?

- Oleinik-Petrovsky-Thom-Milnor technique does not give anything better.

What about a single connected component?

- Oleinik-Petrovsky-Thom-Milnor technique does not give anything better.
- In analogy to the single cell results computational geometry, one might conjecture that the sum of the Betti numbers of a single connected component of a basic semi-algebraic set is bounded by $n^{k-1}O(d)^k$.

What about a single connected component?

- Oleinik-Petrovsky-Thom-Milnor technique does not give anything better.
- In analogy to the single cell results computational geometry, one might conjecture that the sum of the Betti numbers of a single connected component of a basic semi-algebraic set is bounded by $n^{k-1}O(d)^k$.

- It is easy to construct a basic semi-algebraic set such that it has one connected component whose other Betti numbers sum to $\Omega(nd)^{k-1}$.

- Let

$$P_i = (X_k^2 + L_{i,1}^2) \cdots (X_k^2 + L_{i,\lfloor d/2 \rfloor}^2) - \epsilon,$$

where the $L_{ij} \in R[X_1, \ldots, X_{k-1}]$ are generic linear polynomials and $\epsilon > 0$ and sufficiently small. The set S defined by $P_1 \geq 0, \ldots, P_s \geq 0$ has one connected component with $\sum_i \beta_i(S) = \Omega(nd)^{k-1}$.

New Results

Theorem 1. *(B98) Let* C *be a* k*-dimensional cell in an arrangement of* n *surface patches* $S_1, \ldots, S_n$ *in* R^k*. Then the combinatorial complexity of* C *is bounded by* $O(n^{k-1+\epsilon})$ *for every* $\epsilon > 0$*.*

New Results

Theorem 2. *(B98) Let* $C_1, \ldots, C_m \subset R^k$ *be* m *different connected components of a basic semi-algebraic set defined by* $P_1 \geq 0, \ldots, P_n \geq 0$, *with the degrees of the polynomials* P_i *bounded by* d. *Then* $\sum_{i,j} \beta_i(C_j)$ *is bounded by* $m + \binom{n}{k-1} O(d)^k$.

- Proof used Morse theory for stratified spaces.

Different Bounds for Different Betti Numbers

- Consider the union of n compact, convex, s.a. sets in R^k. The *nerve lemma* gives us a bound on the individual Betti numbers of the union.

Different Bounds for Different Betti Numbers

- Consider the union of n compact, convex, s.a. sets in R^k. The *nerve lemma* gives us a bound on the individual Betti numbers of the union.

- The homology groups of the union is isomorphic to the homology groups of the nerve complex. The nerve complex has n vertices and thus the i-th Betti number is bounded by $\binom{n}{i+1}$.

Different Bounds for Different Betti Numbers

- Consider the union of n compact, convex, s.a. sets in R^k. The *nerve lemma* gives us a bound on the individual Betti numbers of the union.
- The homology groups of the union is isomorphic to the homology groups of the nerve complex. The nerve complex has n vertices and thus the i-th Betti number is bounded by $\binom{n}{i+1}$.

- What if the intersections are not acyclic but have bounded topology ?

The Nerve Complex

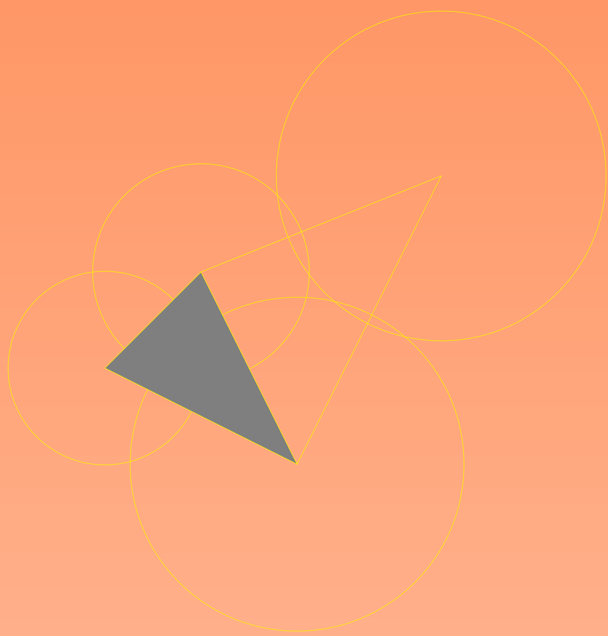


Figure 1: The nerve complex of a union of disks

Betti numbers for union

Theorem 3. *Let* $S \subset R^k$ *be the set defined by the disjunction of* n *inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0,$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

Then,

$$\beta_i(S) \leq \binom{n}{i+1} O(d)^k.$$

Betti numbers for intersections

Theorem 4. *Let $S \subset R^k$ be the set defined by the conjunction of n inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0,$$

$$deg(P_i) \leq d, 1 \leq i \leq n.$$

Then,

$$\beta_i(S) \leq \binom{n}{k-i} O(d)^k.$$

Sets defined by Quadratic Inequalities

- Let $S \subset R^k$ be defined by

$$P_1 \geq 0, \cdots, P_n \geq 0,$$

$$\deg(P_i) \leq 2, 1 \leq i \leq n.$$

Sets defined by Quadratic Inequalities

- Let $S \subset R^k$ be defined by

$$P_1 \geq 0, \cdots, P_n \geq 0,$$

$$\deg(P_i) \leq 2, 1 \leq i \leq n.$$

- They arise in many applications e.g. as the configuration space of sets of points with pair-wise distance constraints etc.

Sets defined by Quadratic Inequalities

- Let $S \subset R^k$ be defined by

$$P_1 \geq 0, \cdots, P_n \geq 0,$$

$$\deg(P_i) \leq 2, 1 \leq i \leq n.$$

- They arise in many applications e.g. as the configuration space of sets of points with pair-wise distance constraints etc.

- Can be topologically quite complicated. If S is defined by

$$X_1(X_1 - 1) \geq 0, \ldots, X_k(X_k - 1) \geq 0,$$

then clearly $\beta_0(S) = 2^k$ (exponential in the dimension).

But ...

Theorem 5. *Let* ℓ *be any fixed number and let* $S \subset R^k$ *be defined by*

$$P_1 \geq 0, \ldots, P_n \geq 0$$

with $\deg(P_i) \leq 2$. *Then,*

$$\beta_{k-\ell}(S) \leq n^{\ell} k^{O(\ell)}.$$

Note that this bound is polynomial in the dimension.

One word about the proofs

The proofs use the spectral sequence associated with the Mayer-Vietoris double complex.