Algebra 557: Week 5

1 Associated Primes and Primary Decomposition

Definition 1. *Let M be an A-module. The set of associated primes of M is defined by*

$$\mathrm{Ass}(M) = \{\mathfrak{p} \in \mathrm{Spec}\, A \mid \exists x \in M, x \neq 0, \text{ with } \mathfrak{p} = \mathrm{ann}(x)\}.$$

Remark 2. It is easy to see that $\mathfrak{p} \in \mathrm{Ass}(M)$ if and only if M has a submodule isomorphic to $A/\mathfrak{p}$ (namely the submodules Ax with $\mathfrak{p} = \mathrm{ann}(x)$).

Example 3. Let $A = \mathbb{Z}$, and $m = p_1^{e_1} \cdots p_r^{e_r}$. Then

$$\mathrm{Ass}(\mathbb{Z}/(m)) = \{p_1, \ldots, p_r\},$$

$$\mathrm{Ass}(\mathbb{Z}) = \{0\},$$

$$\mathrm{Ass}(0) = \emptyset.$$

Proposition 4. *For every $\mathfrak{p} \in \mathrm{Spec}\, A$, and submodule M of $A/\mathfrak{p}$, $M \neq 0$,*

$$\mathrm{Ass}(M) = \{\mathfrak{p}\}.$$

Proof. For every $m = x + \mathfrak{p} \in M$, with $m \neq 0$ (i.e. $x \notin \mathfrak{p}$), we have $ax \in \mathfrak{p} \Leftrightarrow a \in \mathfrak{p}$, since $\mathfrak{p}$ is a prime ideal. Thus, $\mathrm{ann}(m) = \mathfrak{p}$. ☐

Proposition 5. *Maximal elements of the set of ideals $(\mathrm{ann}(x))_{x \in M, x \neq 0}$ belong to $\mathrm{Ass}(M)$.*

Proof. It suffices to prove that if $\mathrm{ann}(x)$ is a maximal element then it is a prime ideal. Suppose that $a \notin \mathrm{ann}(x)$, $ab \in \mathrm{ann}(x)$. We have that $a \in \mathrm{ann}(bx) \supset \mathrm{ann}(x)$. But this contradicts the maximality of $\mathrm{ann}(x)$ unless $bx = 0$. Hence, $b \in \mathrm{ann}(x)$ proving that $\mathrm{ann}(x)$ is prime. ☐

Corollary 6. *If A is Noetherian and M an A-module with $M \neq 0$, then $\mathrm{Ass}(M) \neq 0$.*

Proof. If $x \in M, x \neq 0$ then $\mathrm{ann}(x) \neq A$. This shows that the set $(\mathrm{ann}(x))_{x \in M, x \neq 0}$ is not empty and by the maximal property of Noetherian rings there exists a maximal element in the set. Now apply previous proposition. ☐

Corollary 7. *If A is Noetherian and M an A-module, then for any $a \in A$ the homothety by a on M is injective if and only if $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$.*

Proof. One direction is clear. In the other direction suppose that $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$. If $ax = 0$ with $x \neq 0$, then $a \in \operatorname{ann}(x)$. Consider the submodule $Ax \subset M$. Then there exists $\mathfrak{p} \in \operatorname{Ass}(Ax)$. Then $\mathfrak{p} = \operatorname{ann}(bx)$. But then $a \in \operatorname{ann}(x) \subset \operatorname{ann}(bx) = \mathfrak{p}$. □

Corollary 8. *The set of divisors of zero in a Noetherian ring is the union of its associated primes.*

Proof. Clear from the above as the homothety by $0 \neq a \in A$ is an injection if and only if a is not a divisor of zero. □

Proposition 9. *Let $(M_i)_{i\in I}$ be a family of submodules of M with $M = \cup_{i\in I} M_i$. Then*

$$\operatorname{Ass}(M) = \bigcup_{i\in I} \operatorname{Ass}(M_i).$$

Proof. Clear. □

Proposition 10. *Let M be an A-module and $N \subset M$ a sub-module. Then*

$$\operatorname{Ass}(N) \subset \operatorname{Ass}(M) \subset \operatorname{Ass}(N) \cup \operatorname{Ass}(M/N).$$

Proof. The first inclusion is clear. Now suppose $\mathfrak{p} \in \operatorname{Ass}(M)$. Then M contains a submodule E isomorphic to $A/\mathfrak{p}$. Let $F = E \cap N$. If $F \neq 0$, then by Proposition 4 we have that $\mathfrak{p} \in \operatorname{Ass}(F) \subset \operatorname{Ass}(N)$. Otherwise, $F = 0$ and E descends to an isomorphic submodule of M/N, and we have that $\mathfrak{p} \in \operatorname{Ass}(M/N)$. □

Corollary 11. *If M is the direct sum of the family $(M_i)_{i\in I}$ then*

$$\operatorname{Ass}(M) = \bigcup_{i\in I} \operatorname{Ass}(M_i).$$

Proof. Firstly it can be reduced to the case when I is finite using Proposition 9, since M is the union of the direct sums of the finite subsets of the family $(M_i)_{i\in I}$. Now apply induction and the previous proposition. □

Proposition 12. *Let M be an A-module and Φ a subset of $\operatorname{Ass}(M)$. Then there exists a sub-module $N \subset M$, such that $\operatorname{Ass}(N) = \operatorname{Ass}(M) - \Phi$, and $\operatorname{Ass}(M/N) = \Phi$.*

Proof. Let $\mathfrak{C}$ be the set of sub-modules P of M such that $\operatorname{Ass}(P) \subset \operatorname{Ass}(M) - \Phi$. Then $\mathfrak{C}$ is not empty since $\operatorname{Ass}(0) = \emptyset \subset \operatorname{Ass}(M) - \Phi$, and $\mathfrak{C}$ is inductively ordered by inclusion. Let N be a maximal element of $\mathfrak{C}$. We now prove that $\operatorname{Ass}(M/N) \subset \Phi$, which will imply the proposition (using Proposition 10). Suppose $\mathfrak{p} \in \operatorname{Ass}(M/N)$. Then M/N contains a submodule F/N isomorphic to A/$\mathfrak{p}$. Then, $\operatorname{Ass}(F) \subset \operatorname{Ass}(N) \cup \{\mathfrak{p}\}$. By maximality of N, $\mathfrak{p} \notin \operatorname{Ass}(M) - \Phi$ and $\mathfrak{p} \in \operatorname{Ass}(F) \subset \operatorname{Ass}(M)$. Thus, $\mathfrak{p} \in \Phi$, and $\operatorname{Ass}(M/N) \subset \Phi$. □

Remark 13. Notice that no Noetherian assumption was needed in the previous proposition.

1.1 Relations between Ass(M) and Supp(M).

Theorem 14. *Every prime ideal $\mathfrak{p}$ containing an element of* Ass(M) *belongs to* Supp(M). *Conversely, if A is Noetherian, then every element of* Supp(M) *contains an element of* Ass(M).

Proof. Let $\mathfrak{p} \in \operatorname{Spec} A$ contain an element $\mathfrak{q} = \operatorname{ann}(x) \in \operatorname{Ass}(M)$. Then clearly $\mathfrak{q} \cap A - \mathfrak{p} = \emptyset$. The prime ideals of the local ring $A_{\mathfrak{p}}$ are in bijection with the primes of A not meeting $A - \mathfrak{p}$. Thus $\mathfrak{q} A_{\mathfrak{p}}$ is a prime ideal of $A_{\mathfrak{p}}$, and is equal to $\operatorname{ann}(\bar{x})$, where $\bar{x}$ is the image of x under the canonical homomorphism $\phi: M \to M_{\mathfrak{p}}$. Thus, $\mathfrak{q} A_{\mathfrak{p}} \in \operatorname{Ass}(M_{\mathfrak{p}})$, whence $M_{\mathfrak{p}} \neq 0$. Thus, $\mathfrak{p} \in \operatorname{Supp}(M)$.

Now let $\mathfrak{p} \in \operatorname{Supp}(M)$, then $M_{\mathfrak{p}} \neq 0$. Since, A is Noetherian, $A_{\mathfrak{p}}$is also Noetherian, and we have that $\operatorname{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq \emptyset$. Let $\mathfrak{q} A_{\mathfrak{p}} = \operatorname{ann}(\bar{x}) \in \operatorname{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Then, $\mathfrak{q} \cap A - \mathfrak{p} = \emptyset$ (i.e. $\mathfrak{q} \subset \mathfrak{p}$)) and $\mathfrak{q} = \operatorname{ann}(x)$. Hence, $\mathfrak{q} \in \operatorname{Ass}(M)$. □

Corollary 15. *For any A-module M we have that* Ass(M) ⊂ Supp(M). *Moreover, if A is Noetherian we have that* Supp(M) *and* Ass(M) *have the same set of minimal elements.*

Corollary 16. *If A is a Noetherian ring then*

$$\operatorname{nil}(A) = \bigcap_{\mathfrak{p} \in \operatorname{Ass}(A)} \mathfrak{p}.$$

Proof. We have that $\operatorname{Supp} A = V(\operatorname{ann}(A)) = V(0) = \operatorname{Spec} A$. Thus, Ass($A$) includes the set of minimal prime ideals. □

1.2 Finitely generated modules over a Noetherian ring.

In this section A *is assumed to be Noetherian and M a f.g. A-module.*

Theorem 17. *There exists a descending sequence*

$$M_0 = M \supset M_1 \supset M_2 \supset \cdots \supset M_n = 0,$$

where for each $i, 0 \leq i \leq n-1$, $M_i / M_{i+1} \cong A / \mathfrak{p}_i$, $\mathfrak{p}_i \in \operatorname{Spec} A$.

Proof. Let $\mathfrak{C}$ be the collection of submodules of M satisfying the conclusion of the theorem. Clearly, $\mathfrak{C} \neq \emptyset$, since $0 \in \mathfrak{C}$ and since M is Noetherian $\mathfrak{C}$ admits a maximal element (say N). Then M/N is Noetherian and if $M \neq N$, we would have that $\operatorname{Ass}(M/N) \neq \emptyset$. If $\mathfrak{p} \in \operatorname{Ass}(M/N)$, then M/N has a submodule $E = P/N$ isomorphic to $A/\mathfrak{p}$, and this contradicts the maximal character of N. □

Theorem 18. *Let $M_0 = M \supset M_1 \supset M_2 \supset \cdots \supset M_n = 0$ be the descending sequence as in the previous theorem. Then*

1. $\operatorname{Ass}(M) \subset \{\mathfrak{p}_0, \ldots, \mathfrak{p}_{n-1}\} \subset \operatorname{Supp}(M)$;
2. *the minimal elements of the three sets are the same and coincide with the minimal elements of the set of primes containing* ann(M).

In particular, $\mathrm{Ass}(M)$ *is finite.*

Proof. The second half follows from Corollary 15. The first inclusion of the first half follows by induction on i, after noting that $\mathrm{Ass}(M_i) \subset \mathrm{Ass}(M_{i+1}) \cup \mathrm{Ass}(M_i/M_{i+1})$ and $\mathrm{Ass}(M_i/M_{i+1}) = \mathrm{Ass}(A/\mathfrak{p}_i) = \{\mathfrak{p}_i\}$. The second inclusion is proved as follows. $\mathrm{Supp}(M_i/M_{i+1}) = \mathrm{Supp}(A/\mathfrak{p}_i) = V(\mathfrak{p}_i)$. Since $\mathfrak{p}_i \in V(\mathfrak{p}_i)$ we deduce that $\mathfrak{p}_i \in \mathrm{Supp}(M_i/M_{i+1}) \subset \mathrm{Supp}(M_i) \subset \mathrm{Supp}(M)$.

Finally, since $\mathrm{Supp}(M) = V(\mathrm{ann}(M))$, the minimal primes containing $\mathrm{ann}(M)$ must belong to Supp(M) and conversely. □

Proposition 19. *Let* $\mathfrak{a} \subset A$ *be an ideal. The following conditions are equivalent.*

1. *There exists* $x \in M, x \neq 0$*, such that* $\mathfrak{a}x = 0$.
2. *For each* $a \in \mathfrak{a}$*, there exists* $x \in M, x \neq 0$*, such that* $ax = 0$.
3. *There exists* $\mathfrak{p} \in \mathrm{Ass}(M)$ *with* $\mathfrak{a} \subset \mathfrak{p}$.

Proof. Clearly (1) implies (2). Assuming (2), we have by Corollary 7 that each $a \in \mathfrak{a}$ belong to some element $\mathfrak{p}_a \in \mathrm{Ass}(M)$. Thus, $\mathfrak{a} \subset \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$. Since $\mathrm{Ass}(M)$ is finite this implies that $\mathfrak{a}$ is contained in some $\mathfrak{p} \in \mathrm{Ass}(M)$. Thus, (2) implies (3). Finally, if $\mathfrak{a} \subset \mathfrak{p} = \mathrm{ann}(x) \in \mathrm{Ass}(M)$ then $\mathfrak{a}x = 0$, and $x \neq 0$, whence (3) implies (1). □

Proposition 20. *Under the same assumptions as the previous proposition, for there to exists a* $k \geq 0$ *such that* $\mathfrak{a}^k M = 0$*, it is necessary and sufficient that*

$$\mathfrak{a} \subset \bigcap_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}.$$

Proof. Since the minimal primes of $\mathrm{Ass}(M)$ and $\mathrm{Supp}(M)$ are the same, and $\mathrm{Supp}(M) = V(\mathrm{ann}(M))$, the condition $\mathfrak{a} \subset \bigcap_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$ is equivalent to $V(\mathrm{ann}(M)) \subset V(\mathfrak{a})$ which in turn is equivalent to $\mathfrak{a} \subset \sqrt{\mathrm{ann}(M)}$. Since A is Noetherian this last condition is equivalent to saying that there exists a $k \geq 0$ such that $\mathfrak{a}^k \subset \mathrm{ann}(M)$. □

1.3 Primary submodules.

Definition 21. *We call an endomorphism* $u \in \mathrm{Hom}_A(M, M)$ *to be quasi-nilpotent if for each* $x \in M$*, there exists* $n(x) \geq 0$*, such that* $u^{n(x)}x = 0$.

Proposition 22. *Let* A *be a Noetherian ring and* M *an* A*-module. Then for each* $a \in A$*, the homothety by* a *is quasi-nilpotent if and only if* $a \in \bigcap_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$.

Proof. The homothety a_M is quasi-nilpotent if and only if for every $x \in M$, there exists $n(x)$, such that $a^{n(x)}Ax = 0$. Applying the previous proposition to the f.g. module Ax we see that $a \in \bigcap_{\mathfrak{p} \in \mathrm{Ass(Ax)}} \mathfrak{p}$. The proposition follows from the fact

$$\mathrm{Ass}(M) = \bigcup_{x \in M} \mathrm{Ass}(Ax).$$

□

Proposition 23. *Let A be a Noetherian ring and M an A-module. The following are equivalent.*

1. *$\operatorname{Ass}(M)$ consists of a single element $\mathfrak{p}$.*
2. *For each $a \in A$, the homothety a_M is either injective or quasi-nilpotent.*

When the above conditions are satisfied, we have $\operatorname{Ass}(M) = \{\mathfrak{p}\}$ where $\mathfrak{p}$ is the set of a for which a_M is quasi-nilpotent.

Proof. Follows directly from Corollary 7 and Proposition 22. □

Definition 24. *Let N be an A-module and $Q \subset N$ a submodule. We say that Q is $\mathfrak{p}$-primary if the module $M = N/Q$ satisfies the conditions of Proposition 23.*

Example 25. An ideal $\mathfrak{a} \subset A$ to be $\mathfrak{p}$-primary for a prime ideal $\mathfrak{p}$ if $\operatorname{Ass}(A/\mathfrak{a}) = \{\mathfrak{p}\}$. Using condition (2) of Proposition 23 we see that saying $\mathfrak{a}$ is $\mathfrak{p}$-primary is equivalent to saying that $\mathfrak{p} = \sqrt{\mathfrak{a}}$.

Example 26. Let $\mathfrak{a} \subset A$ be an ideal which is contained in a single prime ideal $\mathfrak{m}$ (which must also be maximal). Then $\mathfrak{a}$ is $\mathfrak{m}$-primary.

Example 27. If $\mathfrak{a} \subset A$ is an ideal and $\mathfrak{m}$ a maximal ideal, then $\mathfrak{a}$ is $\mathfrak{m}$-primary if and only if there exists $n \geq 1$, with $\mathfrak{m}^n \subset \mathfrak{a} \subset \mathfrak{m}$.

Example 28. If A is a PID then all the primary ideals are (0) and (p^n) where p is irreducible and $n \geq 0$.

Remark 29. The power of a prime ideal in general need not be primary. Conversely, not all all primary ideals are powers of prime ideals.

Proposition 30. *Let M be a A-module over a Noetherian ring A and $(Q_i)_{i\in I}$ a family of $\mathfrak{p}$-primary submodules. Then $\bigcap_{i\in I} Q_i$ is also $\mathfrak{p}$-primary.*

Proof. Let $N = \bigcap_{i\in I} Q_i$. Then the natural homomorphism

$$M \to \bigoplus_{i\in I} M/Q_i$$

has kernel N, and hence M/N is a sub-module of $\bigoplus_{i\in I} M/Q_i$. Hence,

$$\operatorname{Ass}(M/N) \subset \bigcup_{i\in I} \operatorname{Ass}(M/Q_i) = \{\mathfrak{p}\}.$$

Since, for each i, $Q_i \neq M$, we have $N \neq M$ and thus $\operatorname{Ass}(M/N) \neq \emptyset$. Hence, $\operatorname{Ass}(M/N) = \{\mathfrak{p}\}$ and N is also $\mathfrak{p}$-primary. □

Definition 31. *Let $N \subset M$ be a submodule of M and let*

$$N = \bigcap_{i\in I} Q_i$$

where each Q_i is primary. Then the above expression is called a primary decomposition of N.

Remark 32. To say that $N = \bigcap_{i \in I} Q_i$ is a primary decomposition of N in the module M is equivalent to saying that $0 = \bigcap_{i \in I} Q_i / N$ is a primary decomposition of 0 in M / N.

Theorem 33. *Let M be a f.g. module over a Noetherian ring A and let $N \subset M$ be a submodule of M. Then there exists a primary decomposition*

$$N = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(M/N)} Q(\mathfrak{p})$$

where each $Q(\mathfrak{p})$ is a $\mathfrak{p}$-primary submodule of M.

Proof. Replacing M by M/N (cf. Remark 32) we can assume without loss of generality that $N = 0$. Now by Theorem 18 we have $\mathrm{Ass}(M)$ is finite. For each $\mathfrak{p} \in \mathrm{Ass}(M)$, by Proposition 12 there exists a submodule $Q(\mathfrak{p})$ of M, such that $\mathrm{Ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\}$ and $\mathrm{Ass}(Q(\mathfrak{p})) = \mathrm{Ass}(M) - \{\mathfrak{p}\}$. Let $P = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(M)} Q(\mathfrak{p})$. Then, $\mathrm{Ass}(P) \subset \mathrm{Ass}(Q(\mathfrak{p}))$ for each $\mathfrak{p} \in \mathrm{Ass}(M)$. Hence, $\mathrm{Ass}(P) = \emptyset$ and thus $P = 0$. □

Remark 34. The proof of Theorem 33 is non-constructive since the proof of Proposition 12 is non-constructive. In particular cases, constructive proofs giving algorithms for computing primary decompositions exist and are implemented in packages such as Macaulay2, Singular etc.

Definition 35. *A primary decomposition $N = \bigcap_{i \in I} Q_i$ is called reduced if it satisfies*

a) *for any subset $J \subset I, J \neq I$, we have that $\bigcap_{j \in J} Q_j \neq N$;*

b) *if $\{\mathfrak{p}_i\} = \mathrm{Ass}(M/Q)$, then $\mathfrak{p}_i \neq \mathfrak{p}_j$ if $i \neq j$.*

Theorem 36. *Let M be a f.g. module over a Noetherian ring A and let $N \subset M$ be a submodule of M. Then there exists a reduced primary decomposition of N.*

Proof. By virtue of Theorem 33 there exists a primary decomposition

$$N = \bigcap_{i \in I} Q_i$$

where each Q_i is a $\mathfrak{p}_i$-primary submodule of M for some $\mathfrak{p}_i \in \mathrm{Ass}(M/N)$. We can also assume that for each subset $J \subset I$, $J \neq I$, we have that $N \neq \bigcap_{j \in J} Q_j$. Now for each $\mathfrak{p} \in \mathrm{Ass}(M/N)$ let

$$H(\mathfrak{p}) = \{i \in I \mid \mathrm{Ass}(M/Q_i) = \{\mathfrak{p}\}\}.$$

Then by Proposition 30 we have that $Q(\mathfrak{p}) = \bigcap_{i \in H(\mathfrak{p})} Q_i$ is also a $\mathfrak{p}$-primary submodule and the decomposition

$$N = \bigcap_{\mathfrak{p} \in \mathrm{Ass}\,(M/N)} Q(\mathfrak{p})$$

is clearly a reduced primary decomposition. □

1.4 Uniqueness properties of primary decomposition.

Theorem 37. *Let M be a f.g. module over a Noetherian ring A and $N \subset M$ a submodule. For a primary decomposition $N = \bigcap_{i \in I} Q_i$, with $\mathrm{Ass}(M/Q_i) = \{\mathfrak{p}_i\}$, to be reduced, it is necessary and sufficient that the $\mathfrak{p}_i$'s be distinct and belong to $\mathrm{Ass}(M/N)$. In this case we also have*

1. $\mathrm{Ass}(M/N) = \bigcup_{i \in I} \{\mathfrak{p}_i\}$, *and*
2. $\mathrm{Ass}(Q_i/N) = \bigcup_{j \in I, j \neq i} \{\mathfrak{p}_j\}$.

Proof. Suppose that the conditions of the theorem are satisfied. We show that the given primary decomposition $N = \bigcap_{i \in I} Q_i$ is reduced. Let $P_i = \bigcap_{j \in I, j \neq i} Q_i$. If $N = P_i$, then M/N is isomorphic to a submodule of $\bigoplus_{j \neq i} M/Q_i$ implying that $\mathrm{Ass}(M/N) \subset \bigcup_{j \in I, j \neq i} \{\mathfrak{p}_j\}$. Since the $\mathfrak{p}_j$'s are assumed to be distinct this would imply that $\mathfrak{p}_i \notin \mathrm{Ass}(M/N)$ contradicting the assumption. Thus, $N \neq P_i$ and the decomposition $N = \bigcap_{i \in I} Q_i$ is reduced.

Conversely, suppose that the given decomposition is reduced. We show that each $\mathfrak{p}_i \in \mathrm{Ass}(M/N)$. We have that $N = P_i \cap Q_i$ and $N \subset P_i, N \neq P_i$. Then, $0 \neq P_i/N \cong P_i + Q_i/Q_i \subset M/Q_i$. Hence, $\mathrm{Ass}(P_i/N) \subset \mathrm{Ass}(M/Q_i) = \{\mathfrak{p}_i\}$. Since, $P_i/N \neq 0$, we have $\mathrm{Ass}(P_i/N) = \{\mathfrak{p}_i\}$. Finally, since P_i/N is a sub-module of M/N we have that $\mathrm{Ass}(P_i/N) \subset \mathrm{Ass}(M/N)$, whence $\mathfrak{p}_i \in \mathrm{Ass}(M/N)$.

We now prove the last two statements. We already have that

$$\bigcup_{i \in I} \{\mathfrak{p}_i\} \subset \mathrm{Ass}(M/N).$$

We also have that

$$\mathrm{Ass}(M/N) \subset \bigcup_{i \in I} \mathrm{Ass}(M/Q_i) = \bigcup_{i \in I} \{\mathfrak{p}_i\},$$

which proves (1).

Finally, we have that

$$N = \bigcap_{j \in I, j \neq i} (Q_i \cap Q_j),$$

and N is the kernel of the homomorphism taking $Q_i \longrightarrow \bigoplus_{j \neq i} Q_i/Q_i \cap Q_j$. Thus, Q_i/N is isomorphic to a submodule of $\bigoplus_{j \neq i} Q_i/Q_i \cap Q_j$ and we obtain that

$$\mathrm{Ass}(Q_i/N) \subset \bigcup_{j \neq i} \mathrm{Ass}(Q_i/Q_i \cap Q_j) = \bigcup_{j \neq i} \mathrm{Ass}(Q_i + Q_j/Q_j) \subset \bigcup_{j \neq i} \mathrm{Ass}(M/Q_j).$$

This implies that

$$\mathrm{Ass}\,(Q_i/N) \subset \bigcup_{j \neq i} \{\mathfrak{p}_j\}.$$

We also have that

$$\bigcup_{i \in I} \{\mathfrak{p}_i\} = \mathrm{Ass}(M/N) \subset \mathrm{Ass}(Q_i/N) \cup \mathrm{Ass}(M/Q_i) = \mathrm{Ass}(Q_i/N) \cup \{\mathfrak{p}_i\}.$$

This together with the last equation implies that

$$\mathrm{Ass}\,(Q_i/N) = \bigcup_{j \neq i} \{\mathfrak{p}_j\}.$$

□

Corollary 38. *If* $N = \bigcap_{j \in J} Q_j$ *is a reduced primary decomposition if and only if* $\mathrm{card}\, J = \mathrm{card}\, \mathrm{Ass}(M/N)$.