Week 6: Flatness and Tor

1 Revision of tensor product of modules.

Theorem 1. *Let A be a commutative ring and*

$$E' \xrightarrow{u} E \xrightarrow{v} E'' \longrightarrow 0$$

be an exact sequence of right A-modules and let F be a left A-module. Then the sequence

$$E' \otimes_A F \xrightarrow{u \otimes 1} E \otimes_A F \xrightarrow{v \otimes 1} E'' \otimes_A F \longrightarrow 0$$

is also exact.

Proof. Clearly $\bar{v} = v \otimes 1$ is surjective since v is surjective and the image of $\bar{u} = u \otimes 1$ is contained in $\ker \bar{v}$. Let M be the image of $\bar{u}$. In order to prove exactness we will show that there are A-module homomorphisms

$$f: E \otimes_A F/M \longrightarrow E'' \otimes_A F$$

$$g: E'' \otimes_A F \longrightarrow E \otimes_A F/M$$

such that both $f \circ g$ and $g \circ f$ are identity mappings which establishes a bijection between $E \otimes_A F/M$ and $E'' \otimes_A F$. Also, since f will factor through the canonical homomorphism $\phi: E \otimes_A F/M \longrightarrow E \otimes_A F/\ker \bar{v}$ this would imply that ϕ is an isomorphism and hence $M = \ker \bar{v}$.

Let $\psi: E \otimes_A F \longrightarrow E \otimes_A F/M$ be the canonical homomorphism and let $x'' \in E''$. Since v is a surjection there exists $x \in E$ such that $v(x) = x''$. We define a homomorphism $g: E'' \otimes_A F \longrightarrow E \otimes_A F/M$ by defining $g(x'' \otimes y) = \overline{x \otimes y}$ where x is any element of E with $v(x) = x''$. We now show that this is well defined. Suppose, there exists $x_1, x_2 \in E$, with $v(x_1) = v(x_2) = x''$. Then $x_1 - x_2 \in \ker v = \operatorname{Im} u$. Hence, $x_1 \otimes y - x_2 \otimes y = (x_1 - x_2) \otimes y \in M$ and so $\overline{(x_1 - x_{2)} \otimes y} = \bar{0}$. Thus, g is well defined. Let $f: E \otimes_A F/M \longrightarrow E'' \otimes_A F$ be defined by $\overline{x \otimes y} \mapsto v(x) \otimes y$. Clearly, $f \circ g, g \circ f$ satisfies the required properties and we are done. ☐

Remark 2. Note that in general if E' is a submodule of a right A-module E and $j: E' \hookrightarrow E$ the canonical injection, then for any left A-module F, the canonical mapping

$$j \otimes 1_F: E' \otimes_A F \longrightarrow E \otimes_A F$$

is not necessarily injective. Take for example $A = \mathbb{Z}$, $E = \mathbb{Z}$, $E' = 2\mathbb{Z}$, $F = \mathbb{Z}/2\mathbb{Z}$. Then, $E' \otimes F \cong E \otimes F \cong F$. But under the canonical mapping for any $x' = 2x \in E'$, and $y \in F$, $(2x) \otimes y = x \otimes 2y = x \otimes 0 = 0$.

*So care must be taken to distinguish, for a submodule $E' \subset E$ and an element $x \in E'$, between the element $x \otimes y$ "calculated in $E' \otimes F$" and the element $x \otimes y$ "calculated in $E \otimes F$ " (in other words, the element $j(x) \otimes y$). This care is not necessary if F is a **flat** module which we define in the next section.*

Even though tensoring does not preserve injective homomorphisms in general in the following situation it does.

Lemma 3. *If $v: M' \longrightarrow M$ is injective and $v(M')$ is a direct summand of M, then the homomorphism $1_E \otimes v$ is injective and its image is a direct summand of $E \otimes_A M$.*

Proof. Follows from the following more general proposition taking $F = M = M' \oplus M''$. $\square$

Proposition 4. *Let $E = \bigoplus_{i \in I} E_i$ and $F = \bigoplus_{j \in J} F_j$ be a right (resp. left) A-module. Then there is a canonical isomorphism*

$$g: E \otimes_A F \longrightarrow \bigoplus_{(i,j) \in I \times J} (E_i \otimes_A F_j)$$

defined by

$$g((\oplus_{i \in I} e_i) \otimes (\oplus_{j \in J} f_j)) = \oplus_{(i,j) \in I \times J} e_i \otimes f_j .$$

Proof. Easy. $\square$

2 Flatness.

Definition 5. *Let E be a right A-module and M a left A-module. We say that the module E is M-flat if for every injection $j: M' \longrightarrow M$ the homomorphism $1_E \otimes j$: $E \otimes_A M' \longrightarrow E \otimes_A M$ is also an injection.*

Lemma 6. *For a right A-module E is M-flat if it is necessary and sufficient that for every finitely generated submodule M' of M the canonical homomorphism $1_E \otimes j: E \otimes_A M' \longrightarrow E \otimes_A M$ is an injection.*

Proof. Let N be a submodule of M and let $z = \sum_{i \in I} x_i \otimes y_i \in E \otimes_A N$. Suppose that the image of z under the homomorphism $1_E \otimes j$ is 0 in $E \otimes_A M$. Let $M' \subset M$ be the submodule of M generated by the finite set of elements $(y_i)_{i \in I}$, that is

$$M' = \sum_{i \in I} A y_i.$$

Then canonical injection $1 \otimes_A j: E \otimes_A M' \rightarrow E \otimes_A M$ factors through $E \otimes_A N$. By hypothesis if the image of z is 0 in $E \otimes_A M$, $z = 0$ in $E \otimes_A M'$ and hence in $E \otimes_A N$. $\square$

Proposition 7. *If the right A-module E is M-flat, then it is also N-flat if N is a submodule or a quotient module of M.*

Proof. The case of the submodule is obvious since for any submodule $N' \subset N$, the homomorphism $1_E \otimes j$: $E \otimes_A N' \longrightarrow E \otimes_A M$, factors through $E \otimes_A N$, and hence if the image of $z \in E \otimes_A N'$ is equal to 0 in $E \otimes_A N$, its image must also be equal to 0 in $E \otimes_A M$, which forces $z = 0$ since E is M-flat.

Now suppose that N is a quotient module of M. Hence there exists an exact sequence

$$0 \longrightarrow L \overset{i}{\longrightarrow} M \overset{v}{\longrightarrow} N \longrightarrow 0.$$

Let $N' \subset N$ be a submodule of N and $M' \subset M$ be the submodule $v^{-1}(N')$ of M. We have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \overset{i'}{\longrightarrow} & M' & \overset{v'}{\longrightarrow} & N' & \longrightarrow & 0 \\ & & \downarrow & & q\downarrow & & p\downarrow & & \\ 0 & \longrightarrow & L & \overset{i}{\longrightarrow} & M & \overset{v}{\longrightarrow} & N & \longrightarrow & 0 \end{array}.$$

Tensoring with E we obtain the following diagram.

$$\begin{array}{ccccccc} E\otimes_A L & \overset{1\otimes i'}{\longrightarrow} & E\otimes_A M' & \overset{1\otimes v'}{\longrightarrow} & E\otimes_A N' & \longrightarrow & 0 \\ \mathrm{Id}\downarrow & & 1\otimes q\downarrow & & 1\otimes p\downarrow & & \\ E\otimes_A L & \overset{1\otimes i}{\longrightarrow} & E\otimes_A M & \overset{1\otimes v}{\longrightarrow} & E\otimes_A N & \longrightarrow & 0 \end{array}.$$

The rows of the diagram above are exact and the homomorphism $1 \otimes q$ is injective since E is assumed to be M-flat. By diagram chasing (check this) we obtain that $1 \otimes p$ must be injective also, proving that E is N-flat. □

Proposition 8. *If $M = \bigoplus_{i\in I} M_i$ is a left A-module and E a right A-module which is M_i-flat for each $i \in I$, then E is M-flat.*

Proof. (finite case) First suppose that $M = M_1 \oplus M_2$, and E is M_i-flat for $i = 1, 2$. Let M' be a sub-module of M and let $M_1' = M_1 \cap M'$ and M_2' the image of M' in M_2 under the canonical homomorphism. Then as in the last proposition we have the following commutative diagram whose rows are exact.

$$\begin{array}{ccccccc} E\otimes_A M_1' & \overset{1\otimes i'}{\longrightarrow} & E\otimes_A M' & \overset{1\otimes p'}{\longrightarrow} & E\otimes_A M_2' & \longrightarrow & 0 \\ 1\otimes r\downarrow & & 1\otimes s\downarrow & & 1\otimes t\downarrow & & \\ E\otimes_A M_1 & \overset{1\otimes i}{\longrightarrow} & E\otimes_A M & \overset{1\otimes p}{\longrightarrow} & E\otimes_A M_2 & \longrightarrow & 0 \end{array}.$$

We also have that the first and the third vertical arrows are injective (since E is M_i-flat for $i = 1, 2$ and $1 \otimes i$ is injective by Lemma 3. By diagram chasing we obtain that so is the middle one, proving that E is M-flat.

The proposition follows by induction on $\operatorname{card} I$ in case I is finite.

(infinite case) If I is infinite then each finitely generated sub-module $M' \subset M$ is contained in a direct sum $\bigoplus_{j\in J} M_j$, where $J \subset I$ is finite. Since by hypothesis E is M_j-flat for each $j \in J$, using the finite case proved above, we obtain that E is M''-flat where $M'' = \bigoplus_{j\in J} M_j$. The homomorphism $1 \otimes j\colon E \otimes_A M' \longrightarrow E \otimes_A M$ factors through the $E \otimes_A M''$. We see that both homomorphisms in this factorization, namely $E \otimes_A M' \longrightarrow E \otimes_A M''$ and $E \otimes_A M'' \longrightarrow E \otimes_A M$ are injections (the first since E is M''-flat and the second because of Lemma 3 since M'' is a direct summand of M), and hence $1 \otimes j\colon E \otimes_A M' \longrightarrow E \otimes_A M$ is also an injection. Now apply Lemma 6 to deduce that E is M-flat. □

Theorem 9. *Let E be a right A-module. Then the following are equivalent.*

1. *E is A-flat.*
2. *E is M-flat for every left A-module M.*

3. *For every exact sequence of left A-modules*

$$M' \xrightarrow{u} M \xrightarrow{v} M''$$

the induced sequence

$$E \otimes_A M' \xrightarrow{1 \otimes u} E \otimes_A M \xrightarrow{1 \otimes v} E \otimes_A M''$$

is exact.

Proof. It is clear that 2. implies 1. Since every A-module M is a quotient of a free A-module, we have that 1. implies 2. (using Proposition 8 and Proposition 7). It is also clear that 3. implies 2. (considering the exact sequence $0 \longrightarrow M' \xrightarrow{u} M$). We now prove that 2. implies 3.

Let $M''' \subset M''$ be the image of v. Consider the exact sequence

$$M' \xrightarrow{u} M \xrightarrow{v} M''' \longrightarrow 0.$$

Applying Theorem 1 (right exactness of tensor product functor) we have that the following sequence is also exact.

$$E \otimes_A M' \xrightarrow{1 \otimes u} E \otimes_A M \xrightarrow{1 \otimes v} E \otimes M''' \longrightarrow 0.$$

Using 2. we have that the canonoical homomorphism

$$1 \otimes j \colon E \otimes M''' \longrightarrow E \otimes M''$$

is an injection. Also the homomophism

$$1 \otimes v \colon E \otimes_A M \longrightarrow E \otimes_A M''$$

factors through $E \otimes_A M'''$, and thus kernels of the homomorphisms

$$1 \otimes v \colon E \otimes_A M \longrightarrow E \otimes_A M'',$$

$$1 \otimes v \colon E \otimes_A M \longrightarrow E \otimes_A M'''$$

are the same proving exactness of the sequence

$$E \otimes_A M' \xrightarrow{1 \otimes u} E \otimes_A M \xrightarrow{1 \otimes v} E \otimes_A M''.$$

□

Definition 10. *We call a right A-module to be flat if it satisfies the equivalent conditions of Theorem 9.*

Proposition 11. *Let E be a right A-module. If E is flat then for each $a \in A$, a not a divisor of 0, $x a = 0$ implies that $x = 0$ for all $x \in E$. In particular, if A is a PID, then E is flat if and only if E is torsion-free.*

Proof. Let $h_a \colon A \to A$, $t \mapsto t a$, be the homothety by a. We have that h_a is an injection since a is not a divisor of 0. Since E is flat we have that the homomorphism $1_E \otimes h_a \colon E \otimes_A A \longrightarrow E \otimes_A A \cong E$ is an injection. The image of $x \otimes 1$ under $1 \otimes h_a$ is $x a$ (after identifying $E \otimes_A A$ with E). If $x a = 0$, then $x \otimes 1 = x = 0$ (by the same identification).

If A is a PID, then E is flat if and only if the canonical map $1_E \otimes h_a$: $E \otimes_A A \rightarrow E \otimes_A A$ is injective for each $a \in A$. By the above argument this is true if and only if E is torsion-free. □

Example 12.

1. $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, but $\mathbb{Z}/n\mathbb{Z}$ $(n \geq 2)$ is not flat as a $\mathbb{Z}$-module.
2. If $A = \mathbb{C}\{x\}$, $E = \mathbb{C}$. E is not flat as an A-module.

Example 13. (geometric example) Consider the affine variety V defined by the equation $XY = 0$, and let π: $V \rightarrow k$ be the projection on the X-co-ordinate. Let $A = k[X]$ and $B = k[V] = k[X, Y]/(XY)$ be the corresponding co-ordinate rings. Then π_*: $A \rightarrow B$, makes B into an A-module. Since A is a PID, and B is not torsion-free, by the previous proposition B is not a flat A-module (which is an algebraic reflection of the fact that the dimension of the fibres of π has a discontinuity at 0).

2.1 Flatness of Quotient Modules

Proposition 14. *Let E be a right A-module. Then the following are equivalent.*

1. *E is flat;*
2. *for every exact sequence*
$$0 \longrightarrow G \xrightarrow{u} H \xrightarrow{v} E \longrightarrow 0$$
and every left A-module F, the following sequence
$$0 \longrightarrow G \otimes_A F \xrightarrow{u \otimes 1} H \otimes_A F \xrightarrow{v \otimes 1} E \otimes_A F \longrightarrow 0$$
is exact;
3. *There exists a flat right A-module H and an exact sequence*
$$0 \longrightarrow G \xrightarrow{u} H \xrightarrow{v} E \longrightarrow 0$$
such that the sequence
$$0 \longrightarrow G \otimes_A F \xrightarrow{u \otimes 1} H \otimes_A F \xrightarrow{v \otimes 1} E \otimes_A F \longrightarrow 0$$
is exact for each $F = A/\mathfrak{a}$ where $\mathfrak{a}$ is a f.g. ideal of A.

Proof. We first prove that 1. implies 2. Let F be the quotient of a free module L, i.e. there exists an exact sequence
$$0 \longrightarrow R \longrightarrow L \longrightarrow F \longrightarrow 0.$$
Tensoring with E we have the following diagram.
$$\begin{array}{ccccc} G \otimes_A R & \longrightarrow & H \otimes_A R & \longrightarrow & E \otimes_A R \\ \downarrow & & \downarrow & & \downarrow \\ G \otimes_A L & \longrightarrow & H \otimes_A L & \longrightarrow & E \otimes_A L \\ \downarrow & & \downarrow & & \downarrow \\ G \otimes_A F & \longrightarrow & H \otimes_A F & \longrightarrow & E \otimes_A F \end{array}.$$

Apply the snake lemma to conclude that the first homomorphism in the last row is an injection.

2. implies 3. is clear by considering E as a quotient of a free module H. Finally, to prove 3. implies 1. take for F in the preceding diagram the quotient module $A/\mathfrak{a}$ for a f.g. ideal $\mathfrak{a} \subset A$, $L = A$, $R = \mathfrak{a}$, and conclude using the snake lemma that the first homomorphism in the last column is an injection, proving that E is flat. □

Proposition 15. *Let*

$$0 \longrightarrow E' \xrightarrow{u} E \xrightarrow{v} E'' \longrightarrow 0$$

be an exact sequence of right A-modules and suppose that E'' is flat. Then for E to be flat it is necessary and sufficient that E' is flat.

Proof. Apply previous proposition. □

Remark 16. In the above proposition if E and E' are flat, then it is not necessary that that E'' is flat. Take for example $A = \mathbb{Z}$, $E = \mathbb{Z}$, $E' = 2\mathbb{Z}$. Then $E'' = \mathbb{Z}/2\mathbb{Z}$ is not flat even though E, E' are. Hence, quotient modules of a flat module need not be flat.

Remark 17. A submodule of a flat module need not be flat. Take for example $A = k[X, Y]$, and $\mathfrak{a} = AX + AY$. Then $\mathfrak{a}$ is not flat as an A-module. (The homomorphism $\mathfrak{a} \otimes_A \mathfrak{a} \longrightarrow \mathfrak{a} \otimes_A A = \mathfrak{a}$ is not injective since $0 \neq X \otimes Y - Y \otimes X \in \mathfrak{a} \otimes_A \mathfrak{a}$ is in the kernel.

2.2 Flatness in terms of relations.

Theorem 18. *Let $(e_\lambda)_{\lambda \in L}$ be a family of elements of a right A-module E with finite support, and let $(f_\lambda)_{\lambda \in L}$ be a family of generators of a left A-module F and suppose that*

$$\sum_{\lambda \in L} e_\lambda \otimes f_\lambda = 0 \in E \otimes_A F.$$

Then there exists a finite family of elements $(x_j)_{j \in J}$ of elements of E and for each $j \in J$ a family $(a_{j\lambda})_{\lambda \in L}$ of elements of A having finite support, such that

$$e_\lambda = \sum_{j \in J} x_j a_{j\lambda} \quad \text{for each } \lambda \in L, \text{ and}$$

$$\sum_{\lambda \in L} a_{j\lambda} f_\lambda = 0 \quad \text{for each } j \in J.$$

Proof. Let F be the quotient of the free module $A^L = \bigoplus_{\lambda \in L} Au_\lambda$ with kernel of relations R, such that f_λ is the image of u_λ. Then we have an exact sequence

$$0 \longrightarrow R \xrightarrow{i} A^L \xrightarrow{p} F \longrightarrow 0.$$

Tensoring with E we obtain an exact sequence

$$E\otimes_A R\stackrel{1\otimes i}{\longrightarrow}E\otimes_A A^L\stackrel{1\otimes p}{\longrightarrow}E\otimes_A F\longrightarrow 0.$$

Note that we have an isomorphism

$$E\otimes_A A^L\cong\bigoplus_{\lambda\in L} E\otimes_A Au_\lambda.$$

The element $\sum_{\lambda\in L} e_\lambda\otimes u_\lambda\in\ker(1\otimes p)=\operatorname{Im}(1\otimes i)$. Let

$$z=\sum_{j\in J} x_j\otimes r_j\in E\otimes_A R$$

be such that $(1\otimes i)(z)=\sum_{\lambda\in L} e_\lambda\otimes u_\lambda$. Here J is a finite set.

Now for each $j\in J$, let

$$i(r_j)=\sum_{\lambda\in L} a_{j\lambda}u_\lambda.$$

Finally, we have

$$\begin{aligned}(1\otimes i)(z) &= (1\otimes i)\sum_{j\in J} x_j\otimes r_j\\ &= \sum_{j\in J} x_j\otimes i(r_j)\\ &= \sum_{j\in J} x_j\otimes\sum_{\lambda\in L} a_{j\lambda}u_\lambda\\ &= \sum_{\lambda\in L}\left(\sum_{j\in J} x_j a_{j\lambda}\right)\otimes u_\lambda\\ &= \sum_{\lambda\in L} e_\lambda\otimes u_\lambda.\end{aligned}$$

Since the last expression is unique because $E\otimes A^L$ is the direct sum of the submodules $E\otimes Au_\lambda$ we get that

$$e_\lambda = \sum_{j\in J} x_j a_{j\lambda} \quad\text{for each } \lambda\in L.$$

Furthermore, since $p(i(r_j))=0$ for each $j\in J$ we obtain

$$\sum_{\lambda\in L} a_{j\lambda}f_\lambda = 0 \quad\text{for each } j\in J.$$

□

Theorem 19. *A right A-module E is F-flat for a left A-module F, if and only if for every finite family $(e_\lambda)_{\lambda\in L}$, $(f_\lambda)_{\lambda\in L}$ of elements of E and F respectively, with*

$$\sum_{\lambda\in L} e_\lambda\otimes f_\lambda=0\in E\otimes_A F,$$

there exists a finite family of elements $(x_j)_{j\in J}$ *of elements of* E, *and for each* $j \in J$ *a family* $(a_{j\lambda})_{\lambda\in L}$ *of elements of* A *having finite support, such that*

$$e_\lambda = \sum_{j\in J} x_j a_{j\lambda} \quad \text{for each } \lambda \in L, \text{ and}$$

$$\sum_{\lambda\in L} a_{j\lambda} f_\lambda = 0 \quad \text{for each } j \in J.$$

Proof. We have that E is F-flat if and only if for every f.g. submodule $F' \subset F$, the homomorphism $1_E \otimes j\colon E \otimes_A F' \longrightarrow E \otimes_A F$ is injective. Let F' be generated by the finite family $(f_\lambda)_{\lambda\in L}$, and suppose that

$$(1 \otimes j)\left(\sum_{\lambda\in L} e_\lambda \otimes f_\lambda\right) = \sum_{\lambda\in L} e_\lambda \otimes j(f_\lambda) = 0.$$

By Theorem 18 we have that there exists a finite family of elements $(x_j)_{j\in J}$ of elements of E, and for each $j \in J$ a family $(a_{j\lambda})_{\lambda\in L}$ of elements of A having finite support, such that

$$e_\lambda = \sum_{j\in J} x_j a_{j\lambda} \quad \text{for each } \lambda \in L, \text{ and}$$

$$\sum_{\lambda\in L} a_{j\lambda} j(f_\lambda) = \sum_{\lambda\in L} a_{j\lambda} f_\lambda = 0 \quad \text{for each } j \in J.$$

This implies that

$$\sum_{\lambda\in L} e_\lambda \otimes f_\lambda = 0 \in E \otimes_A F'$$

proving that E is F-flat. The converse is clear. □

An immediate corollary is

Corollary 20. *A right* A*-module* E *is flat if and only if for every finite family* $(e_\lambda)_{\lambda\in L}$, $(b_\lambda)_{\lambda\in L}$ *of elements of* E *and* A *respectively, with*

$$\sum_{\lambda\in L} e_\lambda b_\lambda = 0,$$

there exists a finite family of elements $(x_j)_{j\in J}$ *of elements of* E, *and for each* $j \in J$ *a family* $(a_{j\lambda})_{\lambda\in L}$ *of elements of* A *having finite support, such that*

$$e_\lambda = \sum_{j\in J} x_j a_{j\lambda} \quad \text{for each } \lambda \in L, \text{ and}$$

$$\sum_{\lambda\in L} a_{j\lambda} b_\lambda = 0 \quad \text{for each } j \in J.$$

Remark 21. In other words: "every relation amongst $(b_\lambda)_{\lambda\in L}$ with coefficients in E is a linear combination (with coefficients in E) of linear relations amongst the $(b_\lambda)_{\lambda\in L}$ with coefficents in A".

2.3 Faithfully flat modules

Theorem 22. *Let E be a right A-module. Then the following are equivalent*

1. *A sequence of left A-modules*

$$N' \xrightarrow{u} N \xrightarrow{v} N''$$

is exact if and only if the sequence

$$E \otimes_A N' \xrightarrow{1 \otimes u} E \otimes_A N \xrightarrow{1 \otimes v} E \otimes_A N''$$

is exact.

2. *E is flat and for any left A-module N, $E \otimes_A N = 0$ implies that $N = 0$.*

3. *E is flat and for any left A-module homomorphism $v: N \to M$, $1_E \otimes v = 0$ implies that $v = 0$.*

4. *E is flat, and for every maximal ideal $\mathfrak{m} \subset A$, $E \neq E\mathfrak{m}$.*

Definition 23. *A right A-module E is called faithfully flat if it satisfies the equivalent conditions of Theorem 22.*

Proof.

i. 1. implies 2.: E is flat by Proposition. Now suppose that $E \otimes_A N = 0$. Consider the sequence $0 \to N \to 0$. After tensoring with E we obtain an exact sequence $0 \to E \otimes_A N \to 0$ (since the middle term is 0). By hypothesis we must then have that the sequence $0 \to N \to 0$ is exact, or in other words $N = 0$.

ii. 2. implies 3.: Suppose that $1_E \otimes v = 0$ for a module homomorphism $v: N \to M$. Let $I = v(N)$, and let $j: I \to M$ denote the inclusion homomorphism. Then $1_E \otimes v = 0$ implies that $1_E \otimes j = 0$. Since E is flat this implies that $E \otimes_A I = 0$, whence by hypothesis $I = 0$. Hence, $v = 0$.

iii. 3. implies 1.: Since by hypothesis E is flat, clearly the exactness of

$$N' \xrightarrow{u} N \xrightarrow{v} N''$$

implies exactness of

$$E \otimes_A N' \xrightarrow{1 \otimes u} E \otimes_A N \xrightarrow{1 \otimes v} E \otimes_A N''.$$

Conversely, suppose that

$$E \otimes_A N' \xrightarrow{1 \otimes u} E \otimes_A N \xrightarrow{1 \otimes v} E \otimes_A N''$$

is exact. Let $I = \operatorname{Im} u$ and $K = \ker v$. It is easy to see that $I \subset K$. To prove the reverse inclusion consider the exact sequence

$$0 \longrightarrow I \xrightarrow{i} K \xrightarrow{p} K/I \longrightarrow 0.$$

Tensoring with E we obtain the exact sequence (since E is flat)

$$0 \longrightarrow E \otimes_A I \xrightarrow{1_E \otimes i} E \otimes_A K \xrightarrow{1_E \otimes p} E \otimes_A (K/I) \longrightarrow 0.$$

Thus, $E\otimes_A K/E\otimes_A I\cong E\otimes_A (K/I)$ and the former is 0 by the exactness of the sequence $E\otimes_A N'\xrightarrow{1\otimes u}E\otimes_A N\xrightarrow{1\otimes v}E\otimes_A N''$. By hypothesis we get $K/I=0$, proving $I=K$.

iv. 2. implies 4.: We have that $E/E\mathfrak{m}\cong E\otimes_A (A/\mathfrak{m})$. Since $A/\mathfrak{m}\neq 0$, by hypothesis we obtain that $E/E\mathfrak{m}\neq 0$.

v. 4. implies 2.: For any proper ideal $\mathfrak{a}\subset A$, let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{a}$. Then by hypothesis we have that $E/E\mathfrak{m}\cong E\otimes A/\mathfrak{m}\neq 0$ implying that $E\neq E\mathfrak{a}$. Now suppose that $N\neq 0$. Choose a non-zero monogeneous sub-module $N'=An'$ of N. Then $N'\cong A/\mathfrak{a}$ for some ideal $\mathfrak{a}\subset A$. We have $E/E\mathfrak{a}\cong E\otimes_A A/\mathfrak{a}\cong E\otimes_A N'\neq 0$. Since E is flat this implies that $E\otimes_A N\neq 0$.

□

Corollary 24. *If E is a faithfully flat module, then E is a faithful and A-module.*

Proof. Suppose $a\in A$, with $xa=0$ for all $x\in E$. Let $v: A\to A$ be the homothety by a. Then $1_E\otimes v=0$, implying by property 3. of Theorem 22 that $v=0$, and hence $a=0$. □

Corollary 25. *If A is a PID, then for a right A-module E to be faithfully flat it is necessary and sufficient that E is torsion free and $E\neq E\mathfrak{p}$ for each prime ideal $\mathfrak{p}$ of A.*

Proof. Since A is a PID, E is flat if and only if E is torsion free. Now apply Theorem 22 (property 4.). □

Example 26. The $\mathbb{Z}$-module $\mathbb{Q}$ is faithful and flat, but not faithfully flat.