Assignment 1 (due Tues, Sept 9)

Unless stated otherwise A is a commutative ring $\neq 0$ and containing 1.

1. Let $x \in \operatorname{nil}(A)$. Prove that $1+x$ is a unit of A.
2. Prove that $x \in \operatorname{rad}(A)$ if and only if $1+ax$ is a unit for every $a \in A$.
3. Prove that a local ring contains no other idempotent element other than 0, 1.
4. Prove that $f = a_0 + a_1 X + \cdots + a_n X^n \in A[X]$ is nilpotent if and only if each a_i is nilpotent.
5. Prove that $f = a_0 + a_1 X + \cdots + a_n X^n \in A[X]$ is a unit if and only if $a_1, \ldots, a_n$ are nilpotent and a_0 is a unit.
6. Prove that in the ring $A[X]$, the Jacobson radical is equal to the nilradical.
7. Suppose that every ideal of A not contained in the nilradical contains a non-zero idempotent (i.e. an element $e \in A$ s.t. $e^2 = e$). Prove that in this case the Jacobson radical is equal to the nilradical.
8. Suppose that every $x \in A$ satisfy the equation $x^n = x$ for some $n \geq 2$ (depending on x). Prove that every prime ideal of A is maximal.
9. Prove that the set of prime ideals of A has minimal elements with respect to inclusions.
10. Let Σ be the set of ideals of A each of whose elements is a zero divisor of A. Prove that Σ has maximal elements and that each maximal element is a prime ideal. Prove that the set of all zero divisors of A is a union of prime ideals.
11. Let k be a field. Prove that the ring $k[[X]]$ of formal power series is a local ring.
12. Let $I \subset A$ be a f.g. ideal satisfying $I = I^2$. Prove that I is generated by one idempotent element $e \in A$.