Assignment 2 (due Tues, Sept 16)

1. Let k be a field and let $V \subset k^n$ be the set of zeros of a family of polynomials in $k[X_1, \ldots, X_n]$. Prove that V is already the set of zeros of a finite subset of this family.

2. Let A be a local ring such that the maximal ideal $\mathfrak{m}$ is principal and $\cap_{n>0} \mathfrak{m}^n = (0)$. Prove that A is Noetherian and every non-zero ideal of A is a power of $\mathfrak{m}$.

3. Let A be an integral domain with field of fractions K. A *fractional ideal* I of A is an A-submodule I of K such that $I \neq 0$ and $\alpha I \subset A$ for some $0 \neq \alpha \in K$. If I is a fractional ideal then we set $I^{-1} = \{\alpha \in K \mid \alpha I \subset A\}$. In the particular case that $II^{-1} = A$, we say that I is *invertible*. Prove that

   i. If I is a fractional ideal, then I^{-1} is also a fractional ideal and $II^{-1} \subset A$.

   ii. An invertible fractional ideal is f.g. as an A-module.

4. A ring A is *Boolean* if $x^2 = x$ for all $x \in A$. In a Boolean ring A show that

   i. $2x = 0$ for all $x \in A$;

   ii. every prime ideal $\mathfrak{p}$ is maximal;

   iii. every f.g. ideal of A is principal.

5. We say that an ideal I is nilpotent if some power of I is (0). Prove that in an Artinian ring an ideal all of whose elements is nilpotent is itself nilpotent.

6. We say that a ring is *primary* if it contains at most one prime ideal (note that (1) is not a prime ideal for us). Prove that an Artinian ring is a direct sum of Noetherian primary rings and this decomposition is unique.