

Assignment 3: Due Sept 23 (Tues)

1. For each $f \in A$, let X_f denote the complement of $V((f))$ in $X = \text{Spec } A$ (also called a principal open set). Show that
 - i. the sets X_f form a basis of open sets in the Zariski topology on X ;
 - ii. $X_f \cap X_g = X_{fg}$;
 - iii. $X_f = \emptyset \iff f$ is nilpotent;
 - iv. $X_f = X \iff f$ is a unit;
 - v. $X_f = X_g \iff \sqrt{(f)} = \sqrt{(g)}$;
 - vi. X is quasi-compact (i.e. every open cover has a finite subcover);
 - vii. An open subset of X is quasi-compact if and only if it is a finite union of principal open sets.
2. Let $X = \text{Spec } A$. Prove that X is a T_0 -space (i.e. for every distinct points $x, y \in X$, there is a neighborhood of x disjoint from y or else there is a neighborhood of y disjoint from x).
3. A topological space X is said to be irreducible if $X \neq \emptyset$ and if every pair of opens sets of X has a non-empty intersection. Show that $X = \text{Spec } A$ is irreducible if and only if $\text{nil}(A)$ is a prime ideal.
4. Let $\phi: A \rightarrow B$ be a ring homomorphism and let $X = \text{Spec } A$ and $Y = \text{Spec } B$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q}) \in X$. Hence, ϕ induces a map $\phi^*: Y \rightarrow X$. Prove that
 - i. ϕ^* is continuous (in the Zariski topology);
 - ii. If $\mathfrak{a} \subset A$ is an ideal, then $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$;
 - iii. If $\mathfrak{b} \subset B$ is an ideal, then $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$;
 - iv. If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\ker \phi)$ of X ;
 - v. If ϕ is injective then $\phi^*(Y)$ is dense in X ;
 - vi. If $\psi: B \rightarrow C$ is another ring homomorphism then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.
5. Prove the following properties of expansion and contraction of ideals.
 - i. $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = \mathfrak{a}_1^e + \mathfrak{a}_2^e$, $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supset \mathfrak{b}_1^c + \mathfrak{b}_2^c$;
 - ii. $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subset \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$, $(\mathfrak{b}_1 \cap \mathfrak{b}_2)^c = \mathfrak{b}_1^c \cap \mathfrak{b}_2^c$;
 - iii. $(\mathfrak{a}_1 \mathfrak{a}_2)^e = \mathfrak{a}_1^e \mathfrak{a}_2^e$, $(\mathfrak{b}_1 \mathfrak{b}_2)^c \supset \mathfrak{b}_1^c \mathfrak{b}_2^c$;
 - iv. $(\mathfrak{a}_1 : \mathfrak{a}_2)^e \subset \mathfrak{a}_1^e : \mathfrak{a}_2^e$, $(\mathfrak{b}_1 : \mathfrak{b}_2)^c \subset \mathfrak{b}_1^c + \mathfrak{b}_2^c$;
 - v. $(\sqrt{\mathfrak{a}})^e \subset \sqrt{\mathfrak{a}^e}$, $(\sqrt{\mathfrak{b}})^c = \sqrt{\mathfrak{b}^c}$.