Assignment 4: Due Sept 30 (Tues)

1. A multiplicative subset $S \subset A$ is said to be *saturated* if the relation $xy \in S$ implies $x \in S$ and $y \in S$. Prove the following.

   a) For a subset S of A to be saturated it is necessary and sufficient that $A - S$ is a union of prime ideals of A.

   b) Let $S \subset A$ be a multiplicative subset and $\bar{S} = \{a \in A \mid \exists b \in A \text{ with } ab \in S\}$. Then, $\bar{S}$ is the smallest saturated multiplicative subset containing S, $A - \bar{S}$ is the union of prime ideals not meeting S, and for every A-module M, the canonical mapping from $S^{-1}M$ to $\bar{S}^{-1}M$ is injective.

2. Let A be a ring and S a multiplicative subset of A. Prove that the set

$$\mathfrak{n} = \{a \in A \mid sa = 0 \text{ for some } s \in S\}$$

   is an ideal. Let $A_1 = A/\mathfrak{n}$ and S_1 the canonical image of S in A_1. Prove that no element of S_1 is a divisor of 0 in A_1 and that the canonical homomoprphism $S^{-1}A \to S_1^{-1}A_1$ is bijective.

3. Let $S = \mathbb{Z} - \{0\}$, $m \geq 1$, and M the $\mathbb{Z}$-module $\oplus_{n \geq 0} \mathbb{Z}/m^n\mathbb{Z}$. Prove that

   a) $S^{-1}M = 0$ (although M is a faithful $\mathbb{Z}$-module);

   b) the canonical mapping $S^{-1}\mathrm{End}_{\mathbb{Z}}(M) \to \mathrm{End}_{S^{-1}\mathbb{Z}}(S^{-1}M)$ is not injective.

4. Give an example of a $\mathbb{Z}$-module M such that, for $S = \mathbb{Z} - \{0\}$, $\mathrm{ann}(M) = 0$, but $\mathrm{ann}(S^{-1}M) = \mathbb{Q}$.

5. Prove the following properties of Noetherian topological spaces.

   a) Every subspace of a Noetherian space is Noetherian.

   b) Let $(A_i)_{i \in I}$ be a finite covering of a topological space X. If the subspaces A_i of X are Noetherian, so is X.

   c) If X is a Noetherian space, the set of irreducible components of X (and hence the set of connected components of X) is finite.

   d) Deduce from part (c) that the number of minimal primes of a Noetherian ring A is finite.