Assignment 5: Due Oct 7 (Tues)

1. Let A be a Noetherian ring, M a f.g. A-module and $\mathfrak{a} = \mathrm{ann}(M)$.
   a) Show that, if $\mathfrak{a}$ is prime, $\mathfrak{a}$ is the least element of $\mathrm{Ass}(M)$.
   b) Show that every prime ideal of $A/\mathfrak{a}$ is associated with M.
   c) Let $\mathfrak{p}$, $\mathfrak{q}$ be two distinct prime ideals of A such that $\mathfrak{p} \subset \mathfrak{q}$. Show that, if $M = A/\mathfrak{p} \oplus A/\mathfrak{q}$, then $\mathrm{Ass}(A/\mathfrak{a}) \neq \mathrm{Ass}(M)$.
2. Let $A = k[X, Y]$ where k is a field, and let $\mathfrak{a}$ be the maximal ideal $AX + AY$. Show that $\mathrm{Supp}(\mathfrak{a}) = \mathrm{Spec}\,A$ is infinite, while $\mathrm{Ass}(\mathfrak{a}) = 0$.
3. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, M a f.g. A-module and P the submodule of M consisting of the $x \in M$ such that $\mathfrak{a}x = 0$. Show that $\mathrm{Ass}(M/P) \subset \mathrm{Ass}(M)$.
4. Show that in the ring $A = \mathbb{Z}[X]$, the ideal $\mathfrak{m} = 2A + AX$ is maximal and that the ideal $\mathfrak{n} = 4A + AX$ is $\mathfrak{m}$-primary, but is not equal to a power of $\mathfrak{m}$.
5. In the ring $B = \mathbb{Z}[2X, X^2, X^3]$ show that the ideal $\mathfrak{p} = 2BX + BX^2$ is prime, but that $\mathfrak{p}^2$ is *not* $\mathfrak{p}$-primary even though its radical is equal to $\mathfrak{p}$.
6. Let k be a field and $A = k[X, Y, Z]/(Z^2 - XY)$; let x, y, z be the canonoical images of X, Y, Z in A. Show that
   a) $\mathfrak{p} = Ax + Az$ is prime;
   b) $\mathfrak{p}^2$ is not primary;
   c) $\mathfrak{p}^2 = \mathfrak{a} \cap \mathfrak{b}^2$ is a primary decomposition of $\mathfrak{p}^2$, where $\mathfrak{a} = Ax$ and $\mathfrak{b} = Ax + Ay + Az$.
7. Verify the claims in problems 5 and 6 using a computer algebra system (Macaulay2 is recommended).