

## Assignment 6: Due Oct 23 (Thus)

1. Let  $A$  be a ring and let  $A[X]$  denote the polynomial ring. For each ideal  $\mathfrak{a} \subset A$ , let  $\mathfrak{a}[X]$  denote the set of all polynomials in  $A[X]$  with coefficients in  $\mathfrak{a}$ . Prove the following.
  - a) If  $\mathfrak{p}$  is a prime ideal in  $A$ , then  $\mathfrak{p}[X]$  is a prime ideal of  $A[X]$ .
  - b) If  $\mathfrak{q}$  is a  $\mathfrak{p}$ -primary ideal in  $A$ , then  $\mathfrak{q}[X]$  is a  $\mathfrak{p}[X]$ -primary ideal in  $A[X]$ .
  - c) If  $\mathfrak{a} = \bigcap_{i \in I} \mathfrak{q}_i$  is a minimal primary decomposition of an ideal  $\mathfrak{a}$  of  $A$ , then  $\mathfrak{a}[X] = \bigcap_{i \in I} \mathfrak{q}_i[X]$  is a minimal primary decomposition of  $\mathfrak{a}[X]$  in  $A[X]$ .
  - d) If  $\mathfrak{p}$  is a minimal prime ideal of  $A$ , then  $\mathfrak{p}[X]$  is a minimal prime ideal of  $A[X]$ .
2. Let  $k$  be a field and  $A = k[X_1, \dots, X_n]$ . Show that the ideals  $\mathfrak{p}_i = AX_1 + \dots + AX_i$  are prime, and all their powers are primary.
3. For any  $\mathfrak{p} \in \text{Spec } A$ , let  $S_{\mathfrak{p}}(0)$  denote the kernel of the canonical homomorphism  $A \rightarrow A_{\mathfrak{p}}$ . Prove the following.
  - a)  $S_{\mathfrak{p}}(0) \subset \mathfrak{p}$ ;
  - b)  $\sqrt{S_{\mathfrak{p}}(0)} = \mathfrak{p}$  if and only if  $\mathfrak{p}$  is a minimal prime ideal of  $A$ .
  - c) If  $\mathfrak{p} \supset \mathfrak{p}'$  then  $S_{\mathfrak{p}}(0) \subset S_{\mathfrak{p}'}(0)$ .
4. Let  $S \subset A$  be a multiplicatively closed subset. For any ideal  $\mathfrak{a} \subset A$ , let  $S(\mathfrak{a})$  denote the contraction of  $S^{-1}\mathfrak{a}$  in  $A$ . (The ideal  $S(\mathfrak{a})$  is called *the saturation of  $\mathfrak{a}$  with respect to  $S$* ). Prove the following.
  - a)  $S(\mathfrak{a}) \cap S(\mathfrak{b}) = S(\mathfrak{a} \cap \mathfrak{b})$ ;
  - b)  $S(\sqrt{\mathfrak{a}}) = \sqrt{S(\mathfrak{a})}$ ;
  - c)  $S(\mathfrak{a}) = A$  if and only if  $\mathfrak{a} \cap S \neq \emptyset$ ;
  - d)  $S_1(S_2(\mathfrak{a})) = (S_1S_2)(\mathfrak{a})$ .
5. Let  $\mathfrak{p} \in \text{Spec } A$ . The *n-th symbolic power of  $\mathfrak{p}$*  is defined to be the ideal (using the notation of the previous exercise)

$$\mathfrak{p}^{(n)} = S_{\mathfrak{p}}(\mathfrak{p}^n)$$

where  $S_{\mathfrak{p}}$  is the multiplicatively closed subset  $A - \mathfrak{p}$ . Prove the following.

- a)  $\mathfrak{p}^{(n)}$  is a  $\mathfrak{p}$ -primary ideal (recall from last assignment that  $\mathfrak{p}^n$  need not be a primary ideal);
- b) If  $\mathfrak{p}^n$  has a primary decomposition, then  $\mathfrak{p}^{(n)}$  is its  $\mathfrak{p}$ -primary component;
- c)  $\mathfrak{p}^n = \mathfrak{p}^{(n)}$  if and only if  $\mathfrak{p}^n$  is  $\mathfrak{p}$ -primary.