Assignment 9: Due Dec 11 (Thus)

1. Let $A \subset B$ be rings with B integral over A. Prove that

   i. If $x \in B$ is a unit of B, then it is a unit of A;

   ii. The radical of A is the contraction of the radical of B.

2. Let A be a subring of B and C the integral closure of A in B. Prove that $C[X]$ is the integral closure of $A[X]$ in $B[X]$.

3. (Noether Normalization) Let k be an infinite field and A a f.g. k-algebra. Then there exists a finite number of elements $y_1, \ldots, y_r \in A$, which are algebraically independent over k, and such that A is integral over $k[y_1, \ldots, y_r]$.

4. Let M be a f.g. module over a Dedekind domain. Prove that M is flat if and only if it is torsion free (we have proved a similar statement in class for PIDs, but a Dedekind domain need not be a PID).

5. Let $S \subset A$ be a multiplicatively closed subset of a Dedekind domain A. Prove that either $S^{-1}A$ is the field of fractions of A, or else it is a Dedekind domain.

6. (Chinese Remainder Theorem for Dedekind domains) Let A be a Dedekind domain and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals of A and $r_1, \ldots, r_n \geq 0$. Then the homomorphism

$$f: A \longrightarrow \prod_{1 \leq i \leq n} (A/\mathfrak{p}_i^{r_i})$$

   yields an isomorphism of rings

$$A/\prod \mathfrak{p}_i^{r_i} \cong \prod (A/\mathfrak{p}_i^{r_i}).$$

7. Deduce from the previous problem that

   i. A Dedekind domain having a finite number of prime ideals is a PID;

   ii. Every ideal in a Dedekind domain is generated by at most two elements.

8. Let $A = \mathbb{Z}[\sqrt{6}]$ and let P be the ideal $2\mathbb{Z} + \sqrt{6}\,\mathbb{Z}$. Prove the following.

   i. A is a Dedekind domain;

   ii. A is not a PID;

   iii. P is a prime ideal and $P^{-1} = (1/2)P$.