ASSIGNMENT 5. DUE IN CLASS FRI, SEP 29, 2017.

1. Let G be a group and H_1 and H_2 be two subgroups of G. Prove or disprove (by giving a counter-example) the following statements.
   (a) $H_1 \cap H_2$ is a subgroup of G.
   (b) $H_1 \cup H_2$ is a subgroup of G.
2. Let Z_{100}^* be the multiplicative group of units mod 100 (as defined in class). Calculate the inverse of the element [53] (i.e. congruence class of 53) in Z_{100}^*.
3. Let p be a prime. Prove or disprove (by giving a counter-example) the following statements.
   (a) The (additive) group Z_p does not have any subgroup other than $\{e\}$ and Z_p itself.
   (b) The (multiplicative) group Z_p^* does not have any subgroup other than $\{e\}$ and Z_p^* itself.
4. Let G be a group. Let $Z(G)$ be the subset of elements of G which commutes with every element of g. In other words, $Z(g) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$.
   (a) Prove that $Z(G)$ is a subgroup of G. ($Z(g)$ is called the *center* of the group G.)
   (b) Describe the center of Z_n.
   (c) Describe the center of D_8.
5. Let $G = SL(2, \mathbb{R})$, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$.
   (a) Prove that $A, B \in G$.
   (b) Show that the order of A is equal to 4, and that of B is equal to 3.
   (c) Prove that the order of AB is infinite.
   (d) Notice that (as exemplified in this problem) the product of two elements having finite orders in a group can itself have infinite order. But the group G in this example is *not* abelian. Can this phenomenon occur in an abelian group? Prove or disprove.