ASSIGNMENT 9. DUE IN CLASS NOV 3, 2017.

The goal of this multi-part assignment is to give a proof of the existence of p-Sylow subgroups. Recall from class, that if p is a prime number, and G a finite group, and r the largest power of p that divides $|G|$, then a subgroup of G of order p^r is called a p-*Sylow* subgroup of G. For the rest of the assignment p is a fixed prime number.

1. Let G be a finite group, H a normal subgroup of G of order p. Suppose that G/H has a subgroup of order n. Prove that G has a subgroup of order pn. (Hint. Let K be the subgroup of G/H of order n, and $f : G \to G/H$ the canonical surjection. Consider the inverse image of K under f.)
2. Prove that if Z is a finite abelian group, and p divides $|Z|$, then Z contains an element whose order is p. (Hint. Use induction on the order of Z. Start with a non-identity element $z \in Z$, and consider the subgroup Z' generated by z. If this subgroup is equal to Z, then ... Otherwise, if p divides $|Z'|$, you can use the inductive hypothesis. If p does not divide $|Z'|$, then consider the quotient group Z/Z', but be careful about what you conclude from the induction hypothesis in this case.)
3. Let G be a finite group such that p divides $|G|$. Prove that G has a p-Sylow subgroup. (Hint. Use induction on $|G|$. If $|G| = p$, then there is nothing to prove. Otherwise, use the *class equation* discussed in class (unintended pun here). There are two cases. If there exists a stabilizer subgroup G_x, $x \notin Z(G)$, whose order is divisible by p^r (where r is the largest power of p dividing G), then we are done since $|G_x| < |G|$ (why ?). Else, prove that G has a non-trivial center whose order is divisible by p. Apply part (2) to deduce that there exists a subgroup of order p contained in $Z(G)$. This subgroup is a normal subgroup of G (why ?). Now proceed again using induction and part (1).)