ASSIGNMENT 1, DUE SEP 8, 2017

Throughout, A denotes a commutative ring with an identity element, and unless specified otherwise all modules referred to below are A-modules.

1. Let G be a module and E, F two submodules such that $E \subset F$. Prove the following.
   (a) If F is a direct factor of G, F/E is a direct factor of G/E. If E is a direct factor of F, E is a direct factor of G.
   (b) If E is a direct factor of G, then E is a direct factor of F. If F/E is a direct factor of G/E, then F is a direct factor of G.
2. (a) Give examples of two submodules M, N of the $\mathbb{Z}$-module $E = \mathbb{Z} \oplus \mathbb{Z}$ such that M and N are direct factors of E, but $M + N$ is not a direct factor of E.
   (b) Show that the following sequence of $\mathbb{Z}$-module homomorphisms is a short exact sequence, which is not split,
   $$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$
   where the second map is multiplication by 2.
3. Let $(M_i)_{i\in I}$ be a (possibly infinite) family of A-modules and $(N_i \subset M_i)_{i\in I}$ a family of submodules. Prove that the modules
   $$(\bigoplus_i M_i)/(\bigoplus_i N_i)$$
   and
   $$\bigoplus_i (M_i/N_i)$$
   are isomorphic.
4. Recall that an A-module E is called cyclic (or monogeneous) if it is generated by a single element. Prove the following.
   (a) Every simple module M (i.e. a module not having any submodules other than 0 and itself) is cyclic.
   (b) Every sub-module of a cyclic module is isomorphic to a quotient module $\mathfrak{b}/\mathfrak{a}$ where $\mathfrak{a}, \mathfrak{b}$ are two left ideals of A with $\mathfrak{a} \subset \mathfrak{b}$.
   (c) Every quotient module of a cyclic module is cyclic.
   (d) Sub-modules of cyclic modules are not necessarily cyclic.
5. Let E, F be two A-modules and $u : E \to F$ a linear mapping.
   (a) Show that the mapping $(x, y) \mapsto (x, y - u(x))$ of the product module $E \times F$ to itself is an automorphism of $E \times F$.
   (b) Deduce that if there exists a linear mapping $v : F \to E$, and an $a \in E$, such that $v(u(a)) = a$, there exists an automorphism of w of $E \times F$ such that $w(a, 0) = (0, u(a))$.

6. (The five lemma) Consider the following diagram of A-modules and linear maps in which the two rows are exact.

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "f_1"] & M_2 \arrow[r] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & M_4 \arrow[r] \arrow[d, "f_4"] & M_5 \arrow[d, "f_5"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5
\end{tikzcd}$$

Prove (by "diagram chasing") the following.
(a) If f_2 and f_4 are injective, and f_1 is surjective, then f_3 is is injective.
(b) If f_2 and f_4 are are surjective, andf_5 is injective, then f_3 is surjective.
(In particular, if f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3.)

7. In a similar vein to Exercise 6 look up the statement and prove by yourself the "snake lemma" (for A-modules). (The following link to a Youtube clip contains actually the entire proof if you are into 1970's films. "It's my turn".) *You do not need to submit this problem.*