ASSIGNMENT 7, DUE NOV 7, 2017

Throughout, A denotes a commutative ring with an identity element, and unless specified otherwise all modules referred to below are A-modules.

1. Let E be a free A-module of rank n, $(e_i)_{1\leq i\leq n}$ a basis of E, $u \in \mathrm{End}(E)$, and X the matrix of u with respect to the basis $(e_i)_{1\leq i\leq n}$. Prove that $\mathrm{adj}(X)$ is the matrix of the endomorphism $\widetilde{u}$, which is characterized by the property that for all $x, y_2, \ldots, y_n \in E$,
$$\widetilde{u}(x) \wedge y_2 \wedge \cdots \wedge y_n = x \wedge u(y_2) \wedge \cdots \wedge u(y_n).$$
2. Let E be a free A-module of rank n, $(e_i)_{1\leq i\leq n}$ a basis of E, $u, v \in \mathrm{End}(E)$, and X, Y are the matrices of u, v respectively with respect to the basis $(e_i)_{1\leq i\leq n}$. Let $0 \leq p \leq n$.
Prove that:
   (a)
$$(\wedge^p u) \circ (\wedge^p v) = \wedge^p (u \circ v).$$
   (b)
$$(\wedge^p X)(\wedge^p Y) = \wedge^p (XY).$$
3. (Generalization of Cayley-Hamilton). Let χ_A denote the characteristic polynomial of the matrix A with entries in k, and let B be a matrix that commutes with A (i.e. $AB = BA$). Prove that $\chi_A(B) = C(A - B)$, where C is a matrix that commutes with both A and B.
4. The object of this exercise is to introduce the *Pfaffian* associated to an alternating bilinear form on a free module of finite rank. Let E be a free A-module of rank n and $(e_i)_{1\leq i\leq n}$ a basis of E.
   (a) A *skew derivation*, Δ, on the exterior algebra $\bigwedge E$ is a linear map, $\Delta : \bigwedge E \to \bigwedge E$ of degree -1, such that for $\alpha \in \wedge^m E, \beta \in \wedge^n E$, $\Delta(\alpha \wedge \beta) = \Delta(\alpha) \wedge \beta + (-1)^m \alpha \wedge \Delta(\beta)$. Notice that Δ induces a linear map $\wedge^1 E = E \to \wedge^0 E = A$ (in other words an element $f \in E^*$). We say that Δ *extends* f.
   Now let $f \in E^*$. Prove that there exists a unique skew derivation on $\bigwedge E$ which extends f.
   (b) Let $\gamma : E \times E \to A$ be an alternating bilnear map. For each $x \in E$, let Δ_x denote the skew derivation associated to the linear form $y \mapsto \gamma(x, y)$, and let $L_x : \bigwedge E \to \bigwedge E$ denote the endomorphism $L_x(y) = x \wedge y$. Denote by $\Lambda_x = L_x + \Delta_x$. Prove that the mapping $E \to \mathrm{End}(\bigwedge E), x \mapsto \Lambda_x$ is a linear map, and that $\Lambda_x \circ \Lambda_x = 0$ for all $x \in E$.
   (c) Extend the linear map in Part (4b) to a linear map $\bigwedge E \to \mathrm{End}(\bigwedge E)$, and denote the image of $x \in \bigwedge E$ under this map also by Λ_x.
   (d) Prove that for $x_1, \ldots, x_p \in E$,
$$\Lambda_{x_1 \wedge \cdots \wedge x_p} = \Lambda_{x_1} \circ \cdots \circ \Lambda_{x_p}.$$

(e) The following is a theorem that you *do not need* to prove. Let $x_1, \dots, x_p \in E$. Then,

$$\det\left[\gamma(x_i, x_j)_{1 \le i,j \le p}\right] = \omega^2,$$

where ω is the component of degree 0 (and hence an element of A) of the element $\Lambda_{x_1} \circ \cdots \circ \Lambda_{x_p}(1) \in \bigwedge E$. If p is odd, then $\omega = 0$.

(f) Now let $X = (a_{ij})_{1 \le i,j \le n}$ be a skew-symmetric $n \times n$ matrix with entries in A. Let $\gamma(e_i, e_j) = a_{ij}$ (cf. Part (4b)). Then, The *Pfaffian* of the matrix A, denoted $\mathrm{Pf}(A)$, is the component of degree 0 of the element $\Lambda_{e_1} \circ \cdots \circ \Lambda_{e_n}(1) \in \bigwedge E$.

Let $n = 4$, and $X = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix}$ be the general 4×4 skew symmetric matrix. Using the definition of $\mathrm{Pf}(X)$ given above, prove using a direct calculation that

$$\mathrm{Pf}(X) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23},$$

and verify the identity

$$\det(X) = \mathrm{Pf}(X)^2$$

in this case.