ASSIGNMENT 9, DUE WED, NOV 29

Throughout this exercise the entries of the matrices belong to a field k which you can assume to be algebraically closed.

1. Let A and B be square matrices of order n, such that $\mathrm{Tr}(A^m) = \mathrm{Tr}(B^m)$ for $m = 1, \ldots, n$. Prove that the eigenvalues of A and B coincide.
2. Let A be an invertible matrix of order n whose minimal polynomial q_A equals its characteristic polynomial. Prove that the minimal polynomial of A^{-1} is given by $q_A(0)^{-1}X^n q_A(1/X)$.
3. Let the minimal polynomial of a matrix A be equal to $\prod_i (X - \lambda_i)^{n_i}$. Prove that the minimal polynomial of the matrix $\begin{pmatrix} A & I \\ 0 & A \end{pmatrix}$ is equal to $\prod_i (X - \lambda_i)^{n_i+1}$.
4. (a) Suppose that characteristic polynomial of a matrix A coincides with its minimal polynomial. What can you say about the block structure of its rational canonical form ?
   (b) Prove that for one cyclic block, the characteristic polynomial equals the minimal polynomial.
5. (a) For 2×2 matrices, characterize those matrices whose minimal polynomial *does not* equal its characteristic polynomial.
   (b) Let A be a 3×3 matrix, not of the form λI_3. Prove that A is similar to a matrix whose diagonal is $(0, 0, \mathrm{Tr}(A))$. (Note that each cyclic block in the rational canonical form is of this form, so by the previous problem the statement is true for matrices whose characteristic polynomial coincides with its minimal polynomial.)
   (c) Prove Part (5b) for $n \times n$ matrices by induction on n.