10.5 The Indeterminate Form 0/0

We know that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

provided

$$\lim_{x\to c} f(x) \text{ and } \lim_{x\to c} g(x) \text{ exist}$$

and

$$\lim_{x\to c} g(x) \neq 0.$$

In this section, we come up with rules to deal with the case where

$$\lim_{x\to c} g(x) = 0.$$

Wherever necessary, we assume that f and g have derivatives that are continuous near c.

Theorem 10.5.1 L'Hospitals' Rule for form 0/0 (page 611 of SHE, 9th edn.)
Suppose that

$$\lim_{x\to c} f(x) = 0$$

and

$$\lim_{x\to c} g(x) = 0.$$

If

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = L,$$

then also

$$\lim_{x\to c} \frac{f(x)}{g(x)} = L.$$

Remarks
(a) In other words,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

provided the limit on the right involving the derivatives exists (as a finite real number, or ∞, or -∞).
(b) We only apply this rule when both $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$. It is not (in general) true when these conditions are violated. So we first check whether we need l'Hospital, and then differentiate the numerator f and denominator g, and try compute the limit of f'/g'.
(c) Exactly the same rule works when

$$\lim_{x\to c}$$

is replaced by

$$\lim_{x\to\infty} \text{ or } \lim_{x\to-\infty} \text{ or } \lim_{x\to c-} \text{ or } \lim_{x\to c+}.$$

(d) The proof uses the Cauchy Mean Value Theorem, see SHE, page 613. Here we do only

Proof of Theorem 10.5.1 in a special case

We assume that $c = 0$ and that f' and g' exist and are continuous in an open interval containing 0, and that $g'(0) \neq 0$. Then f and g are continuous at 0, and

$$\begin{aligned} f(0) &= \lim_{x\to 0} f(x) = 0; \\ g(0) &= \lim_{x\to 0} g(x) = 0. \end{aligned}$$

By the Mean Value Theorem, we can write for some a between 0 and x

$$f(x) - f(0) = f'(a)(x-0)$$

and similarly, for some b between 0 and x,

$$g(x) - g(0) = g'(b)(x-0).$$

If x is close enough to 0, then $g'(b)$ will be close to $g'(0)$ (by continuity of g') and so $g'(b) \neq 0$. Then as $f(0) = g(0) = 0$, this gives

$$\frac{f(x)}{g(x)} = \frac{f'(a)\,x}{g'(b)\,x} = \frac{f'(a)}{g'(b)}.$$

Now as $x \to 0$, both $a, b \to 0$, and the assumed continuity of f', g' gives

$$\begin{aligned} \lim_{x\to 0} \frac{f'(x)}{g'(x)} &= \lim_{x\to 0} \frac{f'(a)}{g'(b)} \\ &= \frac{f'(0)}{g'(0)} \\ &= \lim_{x\to 0} \frac{f'(x)}{g'(x)}. \end{aligned}$$

■

Example 1
Find

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\pi - 2x}.$$

Solution
Here

$$f(x) = \cos x$$

and

$$\lim_{x\to\frac{\pi}{2}} \cos x = \cos\frac{\pi}{2} = 0;$$

while

$$g(x) = \pi - 2x$$

and

$$\lim_{x \to \frac{\pi}{2}} (\pi - 2x) = \pi - 2\left(\frac{\pi}{2}\right) = 0.$$

We try l"Hospital:

$$\begin{aligned} \lim_{x \to \frac{\pi}{2}} \frac{f'(x)}{g'(x)} &= \lim_{x \to \frac{\pi}{2}} \frac{-\sin x}{-2} \\ &= \frac{1}{2} \lim_{x \to \frac{\pi}{2}} \sin x \\ &= \frac{1}{2} \sin \frac{\pi}{2} = \frac{1}{2}. \end{aligned}$$

Then also

$$\lim_{x \to \frac{\pi}{2}} \frac{f(x)}{g(x)} = \frac{1}{2},$$

that is,

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\pi - 2x} = \frac{1}{2}.$$

Example 2
Find

$$\lim_{x \to 0+} \frac{x}{\sin \sqrt{x}}.$$

Solution
Here

$$f(x) = x \text{ and } g(x) = \sin \sqrt{x}$$

so

$$\lim_{x \to 0+} f(x) = 0 \text{ and } \lim_{x \to 0+} g(x) = 0.$$

So we try l"Hospital:

$$\begin{aligned} \lim_{x \to 0+} \frac{f'(x)}{g'(x)} &= \lim_{x \to 0+} \frac{1}{(\cos \sqrt{x}) \frac{1}{2\sqrt{x}}} \\ &= \lim_{x \to 0+} \frac{2\sqrt{x}}{\cos \sqrt{x}} \\ &= \frac{2(0)}{\cos 0} = 0. \end{aligned}$$

Then also

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = 0$$

that is,

$$\lim_{x \to 0+} \frac{x}{\sin \sqrt{x}} = 0.$$

Important Remark
We have already said you must check that BOTH $\lim_{x\to 0+} f(x) = 0$ and $\lim_{x\to 0+} g(x) = 0$. Here is an example of what can go wrong when one of these is violated. Consider

$$\lim_{x\to 0} \frac{x}{\cos x - \sin x} = \frac{0}{1-0} = 0.$$

Here

$$f(x) = x \to 0, x \to 0$$

but

$$g(x) = \cos x - \sin x \to 1, x \to 0.$$

If we try to apply l"Hospital, then we see

$$\frac{f'(x)}{g'(x)} = \frac{1}{-\sin x - \cos x} \to \frac{1}{-1-0} = -1, x \to 0.$$

Thus in this case,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} \neq \lim_{x\to 0} \frac{f'(x)}{g'(x)}.$$

Example 3 Sometimes apply l'Hospital repeatedly
Let F have two continuous derivatives at a (thus F'' exists and F'' is continuous at a). Prove that

$$\lim_{x\to 0} \frac{F(a+x) - 2F(a) + F(a-x)}{x^2} = F''(a).$$

Solution
We have here as $x \to 0$, that the numerator f satisfies

$$\begin{aligned} f(x) &= F(a+x) - 2F(a) + F(a-x) \\ &\to F(a) - 2F(a) + F(a) = 0. \end{aligned}$$

Also the denominator g satisfies

$$g(x) = x^2 \to 0 \text{ as } x \to 0.$$

So we try l"Hospital:

$$\begin{aligned} \frac{f'(x)}{g'(x)} &= \frac{F'(a+x)\cdot 1 - 0 + F'(a-x)(-1)}{2x} \\ &= \frac{F'(a+x) - F'(a-x)}{2x}. \end{aligned}$$

By l"Hospital,

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)}{g(x)} &= \lim_{x\to 0}\frac{f'(x)}{g'(x)}\\ &= \lim_{x\to 0}\frac{F'(a+x)-F'(a-x)}{2x}\end{aligned}$$

if this limit exists. But

$$\lim_{x\to 0}\left(F'(a+x)-F'(a-x)\right)=F'(a)-F'(a)=0$$

and

$$\lim_{x\to 0}(2x)=0.$$

So we try l'Hospital again:

$$\begin{aligned}\frac{f''(x)}{g''(x)} &= \frac{F''(a+x)(1)-F''(a-x)(-1)}{2}\\ &\to \frac{F''(a)+F''(a)}{2}=F''(a)\ \text{ as } x\to 0.\end{aligned}$$

Then l'Hospital tells us that

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{f''(x)}{g''(x)}=F''(a),$$

that is

$$\lim_{x\to 0}\frac{F(a+x)-2F(a)+F(a-x)}{x^2}=F''(a).$$

Example 4
Find

$$\lim_{n\to\infty}\frac{e^{2/n}-1}{1/n}.$$

Solution
Replace the integer variable n by a real variable x. Thus we try to compute

$$\lim_{x\to\infty}\frac{e^{2/x}-1}{1/x}.$$

Here

$$\begin{aligned}f(x) &= e^{2/x}-1\\ &\to e^0-1=0 \text{ as } x\to\infty\end{aligned}$$

while

$$g(x)=1/x\to 0 \text{ as } x\to\infty.$$

So apply l'Hospital:

$$\begin{aligned}\frac{f'(x)}{g'(x)} &= \frac{e^{2/x}\left(-2/x^2\right)}{\left(-1/x^2\right)} \\ &= 2e^{2/x}\end{aligned}$$

so

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty} 2e^{2/x} = 2e^0 = 2.$$

By l'Hospital,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 2,$$

that is,

$$\lim_{x\to\infty}\frac{e^{2/x}-1}{1/x} = 2$$

and hence

$$\lim_{n\to\infty}\frac{e^{2/n}-1}{1/n} = 2.$$

10.6 The Indeterminate Form ∞/∞; Other Indeterminate Forms

We now consider

$$\lim_{x \to c} \frac{f(x)}{g(x)},$$

where

$$\lim_{x \to c} f(x) = \infty = \lim_{x \to c} g(x)$$

and other indeterminate forms.

Theorem 10.6.1 L'Hospitals' Rule for form 0/0 (page 616 of SHE, 9th edn.)
Suppose that

$$\lim_{x \to c} f(x) = \infty$$

and

$$\lim_{x \to c} g(x) = \infty.$$

If

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L,$$

then also

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L.$$

Remarks
(a) In other words,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided the limit on the right involving the derivatives exists (as a finite real number, or ∞, or -∞).
(b) We only apply this rule when both $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = \infty$. It is not (in general) true when these conditions are violated. So we first check whether we need l'Hospital, and then differentiate the numerator f and denominator g, and try compute the limit of f'/g'.
(c) Exactly the same rule works when

$$\lim_{x \to c}$$

is replaced by

$$\lim_{x \to \infty} \text{ or } \lim_{x \to -\infty} \text{ or } \lim_{x \to c-} \text{ or } \lim_{x \to c+}.$$

Example 1
Let $\alpha > 0$. Show that

$$\lim_{x\to\infty} \frac{\ln x}{x^\alpha} = 0.$$

Solution
Here as $x \to \infty$,

$$f(x) = \ln x \to \infty$$

and

$$g(x) = x^\alpha \to \infty.$$

So try l"Hospital:

$$\frac{f'(x)}{g'(x)} = \frac{1/x}{\alpha x^{\alpha-1}} = \frac{1}{\alpha x^\alpha}$$

and hence

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{1}{\alpha x^\alpha} = 0.$$

By l"Hospital,

$$\lim_{x\to\infty} \frac{\ln x}{x^\alpha} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Example 2 Repeated application
Let $k \geq 1$. Show that

$$\lim_{x\to\infty} \frac{x^k}{e^x} = 0.$$

Solution
Here as $x \to \infty$,

$$f(x) = x^k \to \infty$$

and

$$g(x) = e^x \to \infty.$$

So try l"Hospital:

$$\frac{f'(x)}{g'(x)} = \frac{kx^{k-1}}{e^x}$$

but if $k > 1$, still both $x^{k-1} \to \infty$ and $e^x \to \infty$ as $x \to \infty$. So we keep applying l"Hospital:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^k}{e^x} &= \lim_{x\to\infty} \frac{kx^{k-1}}{e^x} \\
&= \lim_{x\to\infty} \frac{k(k-1)x^{k-2}}{e^x} \\
&= \ldots \\
&= \lim_{x\to\infty} \frac{k(k-1)(k-2)\ldots 1}{e^x} \\
&= k! \lim_{x\to\infty} \frac{1}{e^x} = 0.
\end{aligned}$$

Other Indeterminate Forms

(A) Indeterminates of the Form $0 \cdot \infty$ (page 617 of SHE)

If

$$\lim_{x \to c} f(x) = 0 \text{ and } \lim_{x \to c} g(x) = \infty,$$

then we don't know what is

$$\lim_{x \to c} f(x) \cdot g(x),$$

or if it exists. It could be finite or infinite:

Example

If

$$f(x) = x^2 \text{ and } g(x) = \frac{1}{x^3}$$

then

$$\lim_{x \to 0} f(x) = 0 \text{ and } \lim_{x \to 0} g(x) = \infty$$

while

$$\lim_{x \to 0} f(x)\, g(x) = \lim_{x \to 0} x^2 \frac{1}{x^3} = \lim_{x \to 0} \frac{1}{x} = \infty.$$

On the other hand if

$$f(x) = x^2 \text{ and } g(x) = \frac{1}{x}$$

then

$$\lim_{x \to 0} f(x) = 0 \text{ and } \lim_{x \to 0} g(x) = \infty$$

while

$$\lim_{x \to 0} f(x)\, g(x) = \lim_{x \to 0} x^2 \frac{1}{x} = \lim_{x \to 0} x = 0.$$

So how do we cope with this type of limit, called an indeterminate of form $0 \cdot \infty$? We normally write it in the form

$$f \cdot g = \frac{f}{1/g} \text{ (form } \frac{0}{0}\text{)}$$

or

$$f \cdot g = \frac{g}{1/f} \text{ (form } \frac{\infty}{\infty}\text{)}$$

and then apply the version of l"Hospital that we've already used. We have to choose the form that keeps the derivatives simple!.

Example

Find

$$\lim_{x \to 0+} \sqrt{x} \ln x$$

Solution
This has form

$$\lim_{x \to 0+} f(x)\, g(x)$$

where as $x \to 0+$,

$$f(x) = \sqrt{x} \to 0,$$

and

$$g(x) = \ln x \to -\infty.$$

So we rewrite

$$\sqrt{x} \ln x = \frac{\ln x}{1/\sqrt{x}},$$

which has the form $\frac{-\infty}{\infty}$, since as $x \to 0+$,

$$\ln x \to -\infty \text{ and } 1/\sqrt{x} \to \infty.$$

So try l"Hospital:

$$\begin{aligned} \lim_{x \to 0+} \sqrt{x} \ln x &= \lim_{x \to 0+} \frac{\ln x}{1/\sqrt{x}} \\ &= \lim_{x \to 0+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(1/\sqrt{x})} \\ &= \lim_{x \to 0+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} \\ &= \lim_{x \to 0+} (-2)\, x^{1/2} = 0. \end{aligned}$$

Remarks
(a) We could have tried to write

$$\sqrt{x} \ln x = \frac{\sqrt{x}}{1/\ln x},$$

which has form $\frac{0}{0}$, but then the derivatives of $1/\ln x$ get worse and worse, with powers of $\ln x$ appearing. So when doing thse types of limits, we always try to ensure that the derivatives are going to simplify things.
(b) Similarly, if $\alpha > 0$,

$$\lim_{x \to 0+} x^{\alpha} \ln x = 0.$$

(B) Indeterminates of the Form $\infty - \infty$ (page 618 of SHE)

If

$$\lim_{x \to c} f(x) = \infty \text{ and } \lim_{x \to c} g(x) = \infty,$$

we do not know what is

$$\lim_{x \to c} (f(x) - g(x)).$$

This is an indeterminate of the form $\infty - \infty$. We can often convert these to ratios (that is, quotients).

Example
Find

$$\lim_{x \to \frac{\pi}{2}-} (\tan x - \sec x).$$

Solution
Here as $x \to \frac{\pi}{2}-$,

$$\tan x = \frac{\sin x}{\cos x} \to \infty \text{ and } \sec x = \frac{1}{\cos x} \to \infty.$$

So we write

$$\begin{aligned} \tan x - \sec x &= \frac{\sin x}{\cos x} - \frac{1}{\cos x} \\ &= \frac{\sin x - 1}{\cos x}. \end{aligned}$$

Here as $x \to \frac{\pi}{2}-$,

$$\sin x - 1 \to 0 \text{ and } \cos x \to 0,$$

so we have the form $\frac{0}{0}$. By l"Hospital,

$$\begin{aligned} \lim_{x \to \frac{\pi}{2}-} (\tan x - \sec x) &= \lim_{x \to \frac{\pi}{2}-} \frac{\sin x - 1}{\cos x} \\ &= \lim_{x \to \frac{\pi}{2}-} \frac{\frac{d}{dx}[\sin x - 1]}{\frac{d}{dx}[\cos x]} \\ &= \lim_{x \to \frac{\pi}{2}-} \frac{\cos x}{-\sin x} \\ &= \frac{0}{-1} = 0. \end{aligned}$$

(C) Indeterminates of the Form 0^0 or 1^∞ or ∞^0 (page 619 of SHE)

In these types of limits, we have $f(x)^{g(x)}$ for some functions f and g. We can take logs:

$$\ln f(x)^{g(x)} = g(x) \ln f(x)$$

and try use our previous methods. Then afterwards, we take exponentials, since

$$\begin{aligned} f(x)^{g(x)} &= \exp\left(\ln f(x)^{g(x)}\right) \\ &= \exp(g(x) \ln f(x)). \end{aligned}$$

Since exp is continuous, we then use

$$\begin{aligned}\lim_{x\to c} f(x)^{g(x)} &= \lim_{x\to c} \exp\left(g(x)\ln f(x)\right) \\ &= \exp\left(\lim_{x\to c} g(x)\ln f(x)\right),\end{aligned}$$

if the limit inside the exp exists.

Example 1 Form 0^0
Show that

$$\lim_{x\to 0+} x^x = 1.$$

Solution
Here we have $f(x)^{g(x)}$ with $f(x) = g(x) = x \to 0$ as $x \to 0+$. So take logs:

$$\ln x^x = x\ln x,$$

and this has form $0 \cdot (-\infty)$. So we proceed as for limits of the form $0 \cdot \infty$. Write

$$x\ln x = \frac{\ln x}{1/x}$$

giving form $\frac{-\infty}{\infty}$, to which we apply l"Hospital:

$$\begin{aligned}\lim_{x\to 0+} x\ln x &= \lim_{x\to 0+} \frac{\ln x}{1/x} \\ &= \lim_{x\to 0+} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}1/x} \\ &= \lim_{x\to 0+} \frac{1/x}{-1/x^2} \\ &= \lim_{x\to 0+} (-x) = 0.\end{aligned}$$

Now we go back and take exponentials:

$$\begin{aligned}\lim_{x\to 0+} x^x &= \lim_{x\to 0+} \exp\left(\ln x^x\right) \\ &= \exp\left(\lim_{x\to 0+} x\ln x\right) \\ &= \exp(0) = 1.\end{aligned}$$

Example Form ∞^0
Find

$$\lim_{x\to\infty} (1+x)^{1/\sqrt{x}}.$$

Solution
Here

$$\lim_{x\to\infty}(1+x)=\infty$$

and

$$\lim_{x\to\infty}1/\sqrt{x}=0,$$

so our limit has the form ∞^0. So we take log's:

$$\begin{aligned}\ln(1+x)^{1/\sqrt{x}} &= (1/\sqrt{x})\ln(1+x)\\ &= \frac{\ln(1+x)}{\sqrt{x}}.\end{aligned}$$

This now has from $\frac{\infty}{\infty}$. So we try l"Hospital:

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln(1+x)}{\sqrt{x}} &= \lim_{x\to\infty}\frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}\sqrt{x}}\\ &= \lim_{x\to\infty}\frac{\left(\frac{1}{1+x}\right)}{\left(\frac{1}{2\sqrt{x}}\right)}\\ &= \lim_{x\to\infty}\frac{2\sqrt{x}}{1+x}.\end{aligned}$$

Again this has form $\frac{\infty}{\infty}$, so try l"Hospital again: we continue this as

$$\begin{aligned}&= \lim_{x\to\infty}\frac{\frac{d}{dx}(2\sqrt{x})}{\frac{d}{dx}(1+x)}\\ &= \lim_{x\to\infty}\frac{\left(\frac{1}{\sqrt{x}}\right)}{1}\\ &= \lim_{x\to\infty}\frac{1}{\sqrt{x}}=0.\end{aligned}$$

Thus, we have shown

$$\lim_{x\to\infty}\ln(1+x)^{1/\sqrt{x}}=\lim_{x\to\infty}\frac{\ln(1+x)}{\sqrt{x}}=0.$$

Now go back and take exponentials:

$$\begin{aligned}\lim_{x\to\infty}(1+x)^{1/\sqrt{x}} &= \lim_{x\to\infty}\exp\left(\ln(1+x)^{1/\sqrt{x}}\right)\\ &= \exp\left(\lim_{x\to\infty}\ln(1+x)^{1/\sqrt{x}}\right)\\ &= \exp(0)=1.\end{aligned}$$