# 11.1 Infinite Series

## Sigma Notation

Let $a_0, a_1, a_2, ...$ denote numbers. We write

$$\sum_{k=0}^{n} a_k \text{ for } a_0 + a_1 + a_2 + ... + a_n.$$

Thus we first read below the sigma sign $\Sigma$, and see $k = 0$. We read above the sigma sign and see $n$. Then we substitute $k = 0, k = 1, ...,$ in $a_k$, stopping at $k = n$. We add all the $a_k$. The term $k$ is called the **index of summation.**

More generally, if $n \geq m$, we write

$$\sum_{k=m}^{n} a_k \text{ for } a_m + a_{m+1} + a_{m+2} + ... + a_n.$$

Note that $k$ is a "dummy variable". We could use $i, j, \ell$ and so on. Thus also

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + a_{m+2} + ... + a_n.$$

## Rules for Sigma

(I) The **Sigma of the Sum is the Sum of the Sigmas**

$$\sum_{k=0}^{n} (a_k + b_k) = \sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k.$$

**Proof**

$$\begin{aligned} \sum_{k=0}^{n} (a_k + b_k) &= (a_0 + b_0) + (a_1 + b_1) + ... + (a_n + b_n) \\ &= (a_0 + a_1 + ... + a_n) + (b_0 + b_1 + ... + b_n) \\ &= \sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k. \end{aligned}$$

(II) We can **pull a constant outside a sigma sign:**

$$\sum_{k=0}^{n} (2a_k) = 2\sum_{k=0}^{n} a_k.$$

The key issue here is that the 2 does not depend on the index $k$ of summation. More generally, if $C$ is a real number,

$$\sum_{k=0}^{n}(Ca_k) = C\sum_{k=0}^{n}a_k.$$

**Proof**

$$\begin{aligned}\sum_{k=0}^{n}(Ca_k) &= (Ca_0 + Ca_1 + ... + Ca_n)\\ &= C(a_0 + a_1 + ... + a_n)\\ &= C\sum_{k=0}^{n}a_k.\end{aligned}$$

(III) Sometimes we **change the index of summation**: for example,

$$\begin{aligned}\sum_{k=3}^{n}a_k &= a_3 + a_4 + ... + a_n\\ &= \sum_{j=0}^{n-3}a_{3+j}.\end{aligned}$$

(IV) Sometimes we **add a term that does not depend on the index of summation**: for example,

$$\sum_{k=0}^{n}1 = \underbrace{1+1+1+...+1}_{n+1 \text{ times}} = n.$$

More generally, if $C$ is a constant,

$$\sum_{k=m}^{n}C = \underbrace{C+C+C+...+C}_{n \text{ times}} = (n-m+1)C.$$

# Infinite Series

We can add 3 or 100 numbers. Can we add infinitely many? Suppose we have real numbers

$$a_0, a_1, a_2, ... .$$

We can add more and more of these:

$$s_0 = a_0 = \sum_{k=0}^{0}a_k;$$

$$s_1 = a_0 + a_1 = \sum_{k=0}^{1} a_k;$$

$$s_2 = a_0 + a_1 + a_2 = \sum_{k=0}^{2} a_k;$$

$$\vdots$$

$$s_n = a_0 + a_1 + a_2 + ... + a_n = \sum_{k=0}^{n} a_k.$$

We call $s_n$ the **$n$th partial sum**, and we call $\{s_n\}$ **the sequence of partial sums.**

**Definition 11.1.1 (Page 635 of SHE)**
(I) We call

$$\sum_{k=0}^{\infty} a_k$$

an **infinite series.**
(II) If $\{s_n\}$, the sequence of partial sums, converges to a finite limit $L$, then we say that the series

$$\sum_{k=0}^{\infty} a_k \textbf{ converges to } L.$$

We call $L$ the sum of the series in this case.
(III) If $\{s_n\}$ diverges, we say that the series

$$\sum_{k=0}^{\infty} a_k \textbf{ diverges.}$$

**Example 1**
Test

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)}$$

for convergence.
**Solution**
We use partial fractions:

$$\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2},$$

so

$$\begin{aligned}
s_n &= \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} \\
&= \sum_{k=0}^{n} \left[\frac{1}{k+1} - \frac{1}{k+2}\right] \\
&= \left[1-\frac{1}{2}\right] + \left[\frac{1}{2}-\frac{1}{3}\right] + \left[\frac{1}{3}-\frac{1}{4}\right] + \ldots \left[\frac{1}{n}-\frac{1}{n+1}\right] + \left[\frac{1}{n+1}-\frac{1}{n+2}\right] \\
&= 1 - \frac{1}{n+2}.
\end{aligned}$$

Then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1-\frac{1}{n+2}\right) = 1.$$

That, is, the series

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} \text{ converges to } 1,$$

or, we write

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = 1.$$

**Alternative Method, using Sigma notation**

$$\begin{aligned}
s_n &= \sum_{k=0}^{n} \left[\frac{1}{k+1} - \frac{1}{k+2}\right] \\
&= \sum_{k=0}^{n} \frac{1}{k+1} - \sum_{k=0}^{n} \frac{1}{k+2} \\
&= \sum_{k=0}^{n} \frac{1}{k+1} - \sum_{j=1}^{n+1} \frac{1}{j+1} \\
(\text{index change } j &= k+1 \text{ in second sum}) \\
&= \frac{1}{1} + \sum_{k=1}^{n} \frac{1}{k+1} - \sum_{j=1}^{n} \frac{1}{j+1} - \frac{1}{n+2} \\
&= 1 - \frac{1}{n+2}.
\end{aligned}$$

Now proceed as before.

**Telescopic Series**

The sum $s_n$ is above is an example of **a telescopic series**: if $f$ is a function defined on the integers,

$$\sum_{k=m}^{n} [f(k) - f(k+1)] = f(m) - f(n+1).$$

That is, the sum on the left telescopes down to just two terms.

**Example**
Investigate convergence of the infinite series

$$\sum_{k=0}^{\infty} (-1)^k$$

**Solution**
Here

$$\begin{aligned} s_0 &= \sum_{k=0}^{0} (-1)^k = 1; \\ s_1 &= \sum_{k=0}^{1} (-1)^k = 1 + (-1) = 0; \\ s_2 &= \sum_{k=0}^{2} (-1)^k = 1 - 1 + 1 = 1; \\ s_3 &= \sum_{k=0}^{3} (-1)^k = 1 - 1 + 1 - 1 = 0. \end{aligned}$$

We see that

$$s_n = \begin{cases} 1, & n \text{ even;} \\ 0, & n \text{ odd.} \end{cases}$$

Then $\{s_n\}$ diverges, so the series $\sum_{k=0}^{\infty} (-1)^k$ diverges.

# The Geometric Series

The sequence $\{1, x, x^2, ...\} = \{x^n\}$ is called a geometric progression. We now investigate convergence of the **geometric series**

$$\sum_{k=0}^{\infty} x^k.$$

Recall that

$$|x| < 1 \Rightarrow \lim_{n \to \infty} x^n = 0$$

and

$$|x| > 1 \Rightarrow \{x^n\} \text{ is unbounded.}$$

**Theorem 11.1.2 (Page 637 of SHE) Geometric Series**
(i) If $|x| < 1$, then $\sum_{k=0}^{\infty} x^k$ converges and

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

(ii) If $|x| \geq 1$, then $\sum_{k=0}^{\infty} x^k$ diverges.
**Proof**
The $n$th partial sum is

$$s_n = \sum_{k=0}^{n} x^k = 1 + x + x^2 + ... + x^n.$$

Multiply by $x$ :

$$\begin{aligned} xs_n &= x\left(1 + x + x^2 + ... + x^n\right) \\ &= x + x^2 + x^3 + ... + x^{n+1}. \end{aligned}$$

Then

$$\begin{aligned} & s_n - xs_n \\ = & \left(1 + x + x^2 + ... + x^n\right) - \left(x + x^2 + x^3 + ... + x^{n+1}\right) \\ = & 1 - x^{n+1}. \end{aligned}$$

That is

$$s_n (1 - x) = 1 - x^{n+1}$$

or, as long as $x \neq 1$,

$$s_n = \frac{1 - x^{n+1}}{1 - x}. \tag{1}$$

This is the familiar formula for a finite geometric series. Now we consider two subcases:
**(I)** $|x| < 1$
Here

$$\begin{aligned} \lim_{n \to \infty} s_n &= \lim_{n \to \infty} \frac{1 - x^{n+1}}{1 - x} \\ &= \frac{1 - 0}{1 - x} = \frac{1}{1 - x}. \end{aligned}$$

So in this case, the series converges to $\frac{1}{1-x}$.
**(II)** $|x| \geq 1$

**x = 1**
Here

$$\begin{aligned} s_n &= \sum_{k=0}^{n+1} 1 \\ &= \underbrace{1+1+1+\ldots+1}_{n+1 \text{ times}} \\ &= n+1 \to \infty, n \to \infty. \end{aligned}$$

So for $x = 1$ the geometric series diverges.
**x = −1**
Here

$$\begin{aligned} s_n &= \sum_{k=0}^{n} (-1)^k \\ &= \begin{cases} 1, & n \text{ even;} \\ 0, & n \text{ odd.} \end{cases} \end{aligned}$$

as in the example above. So $\{s_n\}$ diverges and the series diverges.
$|x| > \mathbf{1}$
Here from (1),

$$s_n = \frac{1 - x^{n+1}}{1 - x}$$

is unbounded as $n \to \infty$, since $\{x^n\}$ is. So the series diverges. ■

**Example**
Take $x = \frac{1}{2}$ :

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1 - \frac{1}{2}} = 2.$$

# Some Basic Results

**Theorem 11.1.4 (Page 639 of SHE)**
(1) **The Series of The Sum is the Sum of the Series**
If

$$\sum_{k=0}^{\infty} a_k \text{ converges and } \sum_{k=0}^{\infty} b_k \text{ converges,}$$

then

$$\sum_{k=0}^{\infty} (a_k + b_k) \text{ converges}$$

and

$$\sum_{k=0}^{\infty}(a_k+b_k)=\sum_{k=0}^{\infty}a_k+\sum_{k=0}^{\infty}b_k.$$

**(2) Can Pull a Constant out from a Series**
If $\alpha$ is a real number and

$$\sum_{k=0}^{\infty}a_k \text{ converges,}$$

then

$$\sum_{k=0}^{\infty}(\alpha a_k) \text{ converges}$$

and

$$\sum_{k=0}^{\infty}(\alpha a_k)=\alpha\sum_{k=0}^{\infty}a_k.$$

**Example**

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\frac{3}{2^k}+\left(-\frac{1}{3}\right)^k\right)\\ =\;&\sum_{k=0}^{\infty}\frac{3}{2^k}+\sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k\\ =\;&3\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k+\sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k\\ =\;&3\frac{1}{1-\frac{1}{2}}+\frac{1}{1-(-\frac{1}{3})}\\ =\;&6+\frac{1}{\frac{4}{3}}=6\frac{3}{4}.\end{aligned}$$

Here we have used our result for geometric series,

$$\sum_{k=0}^{\infty}x^k=\frac{1}{1-x}, |x|<1.$$

**Theorem 11.1.5 (page 640 of SHE)**
If

$$\sum_{k=0}^{\infty}a_k$$

converges, then

$$\lim_{k\to\infty}a_k=0.$$

**Proof**
As usual, let

$$s_n = a_0 + a_1 + a_2 + ... + a_n$$

denote the $n$th partial sum. Observe that

$$s_n - s_{n-1} = a_n.$$

By definition of convergence,

$$\lim_{n\to\infty} s_n = L,$$

where $L$ is the sum of the series, that is,

$$L = \sum_{k=0}^{\infty} a_k.$$

But then as $n \to \infty$, both $s_n$ and $s_{n-1}$ converge to $L$, so

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} (s_n - s_{n-1}) \\ &= L - L = 0. \end{aligned}$$

■

It follows that if $a_k \nrightarrow 0$ as $k \to \infty$, then the series cannot converge:

**Theorem 11.1.6 (Page 640 of SHE) Basic Divergence Test**
If

$$a_k \nrightarrow 0 \text{ as } k \to \infty$$

then

$$\sum_{k=0}^{\infty} a_k \text{ diverges.}$$

**Example**
Does

$$\sum_{k=0}^{\infty} (-1)^k \frac{k}{k+1} \text{ converge?}$$

**Solution**
We see that here

$$a_k = (-1)^k \frac{k}{k+1}, k \geq 0.$$

Now

$$\begin{aligned} \lim_{k\to\infty} |a_k| &= \lim_{k\to\infty} \frac{k}{k+1} \\ &= \lim_{k\to\infty} \frac{k}{k\left(1+\frac{1}{k}\right)} \\ &= \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} = \frac{1}{1+0} = 1 \neq 0. \end{aligned}$$

(You could also do this using l' Hospital's rule, showing

$$\lim_{x \to \infty} \frac{x}{x+1} = 1.)$$

Then

$$a_k \nrightarrow 0 \text{ as } k \to \infty$$

so

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1)^k \frac{k}{k+1} \text{ diverges.}$$