11.2 The Integral and Comparison Tests

In this section, we assume that all the terms a_k in our series are nonnegative. Note that then

$$\begin{aligned} s_{n+1} &= a_0 + a_1 + a_2 + ... + a_n + a_{n+1} \\ &= s_n + a_{n+1} \\ &\geq s_n. \end{aligned}$$

That is, as n gets bigger, s_n should also. More precisely, $\{s_n\}$ is a nondecreasing sequence. Then there are only two possible behaviors for the sequence of partial sums: Either $\{s_n\}$ is bounded above and then converges to a finite limit. Or, $\{s_n\}$ diverges to infinity. Thus:

Theorem 11.2.1 (Page 643 of SHE)
Let $a_k \geq 0, k \geq 0$. Then

$$\sum_{k=0}^{\infty} a_k \text{ converges}$$

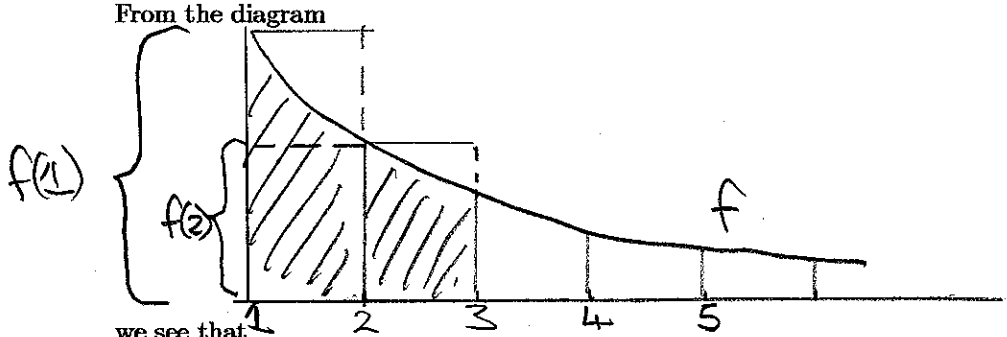
iff the sequence $\{s_n\}$ of partial sums is bounded.

Now we can use improper integrals to study convergence of series:

Theorem 11.2.2 The Integral Test (Page 644 of SHE)
Let f be continuous, positive and decreasing on $[1, \infty)$. Then

$$\sum_{k=1}^{\infty} f(k) \text{ converges iff } \int_1^{\infty} f(x)\,dx \text{ converges.}$$

Idea of Proof
From the diagram



we see that

$$f(1) \cdot 1 \geq \int_1^2 f(x)\,dx \geq f(2) \cdot 1$$

and more generally, for $k \geq 1$,

$$f(k) \cdot 1 \geq \int_k^{k+1} f(x)\, dx \geq f(k+1) \cdot 1.$$

Add these inequalities for $k = 1, 2, \ldots n$:

$$\begin{aligned} & f(1) + f(2) + \ldots + f(n) \\ \geq & \int_1^2 f(x)\, dx + \int_2^3 f(x)\, dx + \ldots + \int_n^{n+1} f(x)\, dx \\ \geq & f(2) + f(3) + \ldots + f(n+1). \end{aligned}$$

That is

$$s_n \geq \int_1^{n+1} f(x)\, dx \geq s_{n+1} - f(1). \quad (1)$$

If the sequence $\{s_n\}$ of partial sums is bounded (above), the same will be true of the sequence

$$\left\{ \int_1^n f(x)\, dx \right\}$$

and then as this is increasing,

$$\lim_{n \to \infty} \int_1^n f(x)\, dx$$

exists and is finite. Since f is decreasing, we can deduce that

$$\int_1^\infty f(x)\, dx = \lim_{b \to \infty} \int_1^b f(x)\, dx$$

exists and is finite. That is, the improper integral

$$\int_1^\infty f(x)\, dx \text{ converges.}$$

Conversely, (1) shows that if the improper integral converges then the sequence $\{s_{n+1}\}$ is bounded above, so the same will be true of $\{s_n\}$. Then by Theorem 11.21.1, the series converges. ■

Remark

We could also sum from $k = 2$ or 3 or some other index. Thus we can also use the integral test in the form

$$\sum_{k=3}^{\infty} f(k) \text{ converges iff } \int_3^\infty f(x)\, dx \text{ converges,}$$

provided of course f is continuous, positive, decreasing on $[3, \infty)$.

Example 11.2.4 (page 646 of SHE) The $p-$Series Converges iff $p > 1$
Let $p > 0$. Show that the p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges iff $p > 1$.
Solution
We see that the series has the form

$$\sum_{k=1}^{\infty} f(k) \text{ where } f(x) = \frac{1}{x^p}.$$

Here f is positive, decreasing, continuous on $[1, \infty)$. By the integral test, the series converges iff

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^p}dx \text{ converges.}$$

In Section 10.7, we saw that this integral converges iff $p > 1$. So the series converges iff $p > 1$.

Remark
An important special case is $p = 1$:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \text{ diverges.}$$

This seris is called the **harmonic series.**

Example
For which $p > 0$ does

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$$

converge?
Solution
Here

$$f(x) = \frac{1}{x(\ln x)^p}$$

is positive, continuous, and decreasing in $[2, \infty)$. (Recall that x and $\ln x$ are positive and increasing there). By the integral test,

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p} = \sum_{k=2}^{\infty} f(k) \text{ converges}$$

iff

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}dx = \int_2^{\infty} f(x)\,dx \text{ converges.}$$

So test the integral for convergence: let $b > 2$. We see that (with the substitution $u = \ln x$)

$$\begin{aligned} & \int_2^b \frac{1}{x(\ln x)^p} dx \\ = & \int_{\ln 2}^{\ln b} \frac{1}{u^p} du \end{aligned}$$

so

$$\begin{aligned} \lim_{b \to \infty} \int_2^b \frac{1}{x(\ln x)^p} dx &= \lim_{b \to \infty} \int_{\ln 2}^{\ln b} \frac{1}{u^p} du \\ &= \int_{\ln 2}^{\infty} \frac{1}{u^p} du, \end{aligned}$$

which converges iff $p > 1$. So,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx \text{ converges iff } p > 1$$

and hence

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p} \text{ converges iff } p > 1.$$

Some Remarks on Notation

(i) Let $j \geq 0$. Note that

$$\sum_{k=0}^{\infty} a_k \text{ converges iff } \sum_{k=j}^{\infty} a_k \text{ converges.}$$

This is because the second series omits only finitely many terms included in the first, and we see that we are excluding

$$\sum_{k=0}^{j-1} a_k = a_0 + a_1 + a_2 + \ldots + a_{j-1}.$$

Then the difference between partial sums for the two series is this fixed number; and consequently it cannot affect convergence. It does affect the value, of course. In fact,

$$\sum_{k=0}^{\infty} a_k - \sum_{k=j}^{\infty} a_k = \sum_{k=0}^{j-1} a_k = a_0 + a_1 + a_2 + \ldots + a_{j-1}$$

provided either series on the left converges.

(ii) Because of this fact, and because we are often interested in convergence of the series, rather than the actual value, we sometimes omit the lower and upper indices of summation. So we write

$$\sum a_k$$

instead of

$$\sum_{k=j}^{\infty} a_k,$$

and talk of $\sum a_k$ converging or diverging. The main point to remember is that $\sum a_k$ is an abbreviation for

$$\sum_{k=j}^{\infty} a_k,$$

for some j.
(iii) Some property of a sequence $\{a_k\}$ is said to hold for k sufficiently large, if it holds for all large enough k. For example, we say

$$a_k \leq \frac{1}{k^2} \text{ for } k \text{ sufficiently large}$$

if there exists an integer m such that

$$a_k \leq \frac{1}{k^2} \text{ for } k \geq m.$$

Theorem 11.2.5 (page 647 of SHE) Basic Comparison Test
Let for all k sufficiently large,

$$a_k \geq 0$$

(I) Suppose that

$$a_k \leq c_k \text{ for } k \text{ sufficiently large}$$

and

$$\sum c_k \text{ converges.}$$

Then

$$\sum a_k \text{ converges.}$$

(II) Suppose that

$$a_k \geq d_k \text{ for } k \text{ sufficiently large}$$

and

$$\sum d_k \text{ diverges.}$$

Then

$$\sum a_k \text{ diverges.}$$

Idea of Proof
(I) The partial sums for $\sum c_k$ will be bounded by Theorem 11.2.1 (because the series converges). Then the partial sums of $\sum a_k$, which are no larger, will also be bounded. By Theorem 11.2.1, $\sum a_k$ will converge.
■

Example
Test

$$\sum \frac{8}{5k^2+3}$$

for convergence.
Solution
The terms look roughly like $\frac{8}{5k^2}$ for k suffucuently large. So use

$$a_k = \frac{8}{5k^2+3} \leq \frac{8}{5k^2} = c_k.$$

Here

$$\sum c_k = \sum \frac{8}{5} k^{-2} = \frac{8}{5} \sum k^{-2}$$

converges (it is a constant times the $p-$series with $p = 2$). The comparison test shows that

$$\sum a_k = \sum \frac{8}{5k^2+3} \text{ converges.}$$

Example
Show that

$$\sum_{k=2}^{\infty} \frac{1}{\ln k}$$

diverges.
Solution
The basic idea we use is that $\ln x$ grows more slowly than x, as $x \to \infty$, so we expect $\frac{1}{\ln k}$ to go to 0 slower than $\frac{1}{k}$. Since $\sum \frac{1}{k}$ diverges ($p-$series with $p = 1$), we then obtain what we need. Let us make this precise: recall from l'Hospital's rule that

$$\lim_{x \to \infty} \frac{\ln x}{x} = 0.$$

Then

$$\lim_{k \to \infty} \frac{\ln k}{k} = 0$$

so for sufficiently large k,

$$\frac{\ln k}{k} < 1.$$

But then for sufficiently large k,

$$\Rightarrow \frac{1}{\ln k} > \frac{1}{k}.$$

Here

$$\sum \frac{1}{k} \text{ diverges}$$

($p-$series with $p = 1$) so the comparison test shows that

$$\sum \frac{1}{\ln k} \text{ diverges.}$$

Sometimes it is easier to compare the terms of two series only for very large indices k, or even as $k \to \infty$. Then the following test proves useful:

Theorem 11.2.6 (p. 649 of SHE) Limit Comparison Test
Let

$$\sum a_k \text{ and } \sum b_k$$

be series with positive terms. Suppose also

$$\lim_{k \to \infty} a_k / b_k = L,$$

where L is a (finite) positive number. Then

$$\sum a_k \text{ converges iff } \sum b_k \text{ converges.}$$

Idea of Proof
Let $0 < \varepsilon < L$. For large enough k,

$$L - \varepsilon < \frac{a_k}{b_k} < L + \varepsilon.$$

Then

$$a_k < (L + \varepsilon) b_k$$

and so if

$$\sum b_k \text{ converges,}$$

the ordinary comparison test gives that

$$\sum a_k \text{ converges.}$$

■

Example
Test the following series for convergence.

$$\sum_{k=2}^{\infty} \frac{1}{5k^{3/2} - 7}$$

Solution
Here

$$a_k = \frac{1}{5k^{3/2} - 7}.$$

We see that for large k, a_k behaves much like

$$\frac{1}{5k^{3/2}}.$$

We also know from p–series, that if

$$b_k = \frac{1}{k^{3/2}},$$

then

$$\sum b_k = \sum \frac{1}{k^{3/2}} \text{ converges.}$$

Thus we should compare our "unknown" a_k to the known p-series b_k. So we use the limit comparison test:

$$\begin{aligned}
\lim_{k\to\infty} a_k/b_k &= \lim_{k\to\infty} \frac{1}{5k^{3/2}-7} / \frac{1}{k^{3/2}} \\
&= \lim_{k\to\infty} \frac{k^{3/2}}{5k^{3/2}-7} \\
&= \lim_{k\to\infty} \left(\frac{k^{3/2}}{k^{3/2}} \frac{1}{5 - 7/k^{3/2}} \right) = \frac{1}{5} > 0.
\end{aligned}$$

By the limit comparison test,

$$\sum b_k \text{ convergent} \Rightarrow \sum a_k \text{ convergent,}$$

that is

$$\sum_{k=2}^{\infty} \frac{1}{5k^{3/2}-7} \text{ converges.}$$

Example
Test for convergence the following series:

$$\sum \frac{5\sqrt{k}+50}{2k\sqrt{k}+18\sqrt{k}}.$$

Solution
Here

$$a_k = \frac{5\sqrt{k}+50}{2k\sqrt{k}+18\sqrt{k}}.$$

We pull out the largest powers of k from the numerator and denominator, to see what a_k behaves like for k large:

$$\begin{aligned}
a_k &= \frac{\sqrt{k}}{k\sqrt{k}} \left(\frac{5+50/\sqrt{k}}{2+18/k} \right) \\
&= \frac{1}{k} \left(\frac{5+50/\sqrt{k}}{2+18/k} \right).
\end{aligned}$$

This suggests that for large k, a_k behaves like $\frac{1}{k}\left(\frac{5}{2}\right)$. So choose

$$b_k = \frac{1}{k},$$

because we recognize this as the term in a p-series, which we know. Next,

$$\begin{aligned}
\lim_{k\to\infty} a_k/b_k &= \lim_{k\to\infty} \left[\frac{1}{k} \left(\frac{5+50/\sqrt{k}}{2+18/k} \right) \right] / \left[\frac{1}{k} \right] \\
&= \lim_{k\to\infty} \left[\frac{5+50/\sqrt{k}}{2+18/k} \right] = \frac{5}{2} > 0.
\end{aligned}$$

Since

$$\sum b_k = \sum \frac{1}{k} \text{ diverges } (p-\text{series with } p=1),$$

so

$$\sum a_k = \sum \frac{5\sqrt{k}+50}{2k\sqrt{k}+18\sqrt{k}} \text{ diverges.}$$

11.3 The Root and Ratio Tests

The root and ratio tests are amongst the most powerful tests. They are both based on comparing a given series to a geometric series $\sum x^k$. Recall that the geometric series converges for $|x| < 1$ and diverges for $|x| \geq 1$. Observe too that

$$\lim_{k\to\infty} \left(x^k\right)^{1/k} = x.$$

Theorem 11.3.1 (Page 653 of SHE) The Root Test
Let $a_k \geq 0$ for all k. Assume that

$$\lim_{k\to\infty} a_k^{1/k} = \rho.$$

(a) If $\rho < 1$, $\sum a_k$ converges.
(b) If $\rho > 1$, $\sum a_k$ diverges.
(If $\rho = 1$, the test is inconclusive).
Proof
(a) We assume $\rho < 1$. Choose μ such that

$$\rho < \mu < 1.$$

For sufficiently large k, we have

$$a_k^{1/k} < \mu$$
$$\Rightarrow a_k < \mu^k.$$

Since $|\mu| = \mu < 1$,

$$\sum \mu^k \text{ is a convergent geometric series.}$$

By the comparison test,

$$\sum a_k \text{ converges.}$$

(b) is similar. ■

Remark
We often use the root test when our terms a_k are of the form something raised to a kth power.

Example
Test for convergence

$$\sum \left(\frac{1}{\ln k}\right)^k.$$

Solution
Here

$$a_k = \left(\frac{1}{\ln k}\right)^k$$

so

$$\lim_{k\to\infty} a_k^{1/k} = \lim_{k\to\infty}\left(\left(\frac{1}{\ln k}\right)^k\right)^{1/k}$$
$$= \lim_{k\to\infty}\frac{1}{\ln k} = 0 < 1.$$

By the root test,

$$\sum a_k = \sum\left(\frac{1}{\ln k}\right)^k \text{ converges.}$$

Example
Test for convergence

$$\sum \frac{k^{100}}{3^k}.$$

Solution
Here

$$a_k = \frac{k^{100}}{3^k}$$

$$\Rightarrow \quad a_k^{1/k} = \left(\frac{k^{100}}{3^k}\right)^{1/k}$$
$$= \frac{k^{100/k}}{3}$$
$$= \frac{\left(k^{1/k}\right)^{100}}{3}.$$

Recall that

$$\lim_{x\to\infty} x^{1/x} = 1 \Rightarrow \lim_{k\to\infty} k^{1/k} = 1.$$

So

$$\lim_{k\to\infty} a_k^{1/k} = \lim_{k\to\infty}\frac{\left(k^{1/k}\right)^{100}}{3} = \frac{1}{3} < 1.$$

By the root test, the series

$$\sum a_k = \sum\frac{k^{100}}{3^k} \text{ converges.}$$

Example 3
Test for convergence

$$\sum k^k.$$

Solution
Here

$$a_k^{1/k} = k \to \infty, k \to \infty.$$

That is

$$\rho = \lim_{k\to\infty} a_k^{1/k} = \infty > 1.$$

By the root test, the series diverges.

The root test works well with powers. The rastio test works well with factorials:

Theorem 11.3.2 (Page 654 of SHE) The Ratio Test
Let $a_k \geq 0$ for all k. Suppose that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lambda.$$

(a) If $\lambda < 1$, then $\sum a_k$ converges.
(b) If $\lambda > 1$, then $\sum a_k$ diverges.
(If $\lambda = 1$, the test is inconclusive).
Proof
One uses the comparison test, showing, very roughly, that a_k is comparable to λ^k. ■

Example
Test for convergence

$$\sum \frac{4^k}{k!}.$$

Solution
Here

$$a_k = \frac{4^k}{k!}$$

so

$$\begin{aligned} a_{k+1}/a_k &= \left[\frac{4^{k+1}}{(k+1)!}\Big/\frac{4^k}{k!}\right] \\ &= 4\frac{k!}{(k+1)!} \\ &= 4\frac{1}{k+1}. \end{aligned}$$

Then

$$\lim_{k\to\infty} a_{k+1}/a_k = \lim_{k\to\infty} \frac{4}{k+1} = 0 < 1.$$

By the ratio test, the series converges.

Example
Test for convergence

$$\sum \frac{(3k)!}{100^k}.$$

Solution
Here

$$a_k = \frac{(3k)!}{100^k}$$

so

$$\begin{aligned} a_{k+1}/a_k &= \frac{(3(k+1))!}{100^{k+1}} \Big/ \frac{(3k)!}{100^k} \\ &= \frac{(3k+3)!}{(3k)!} \frac{100^k}{100^{k+1}} \\ &= (3k+3)(3k+2)(3k+1)\frac{1}{100} \\ &\to \infty, k \to \infty. \end{aligned}$$

That is,

$$\lim_{k \to \infty} a_{k+1}/a_k = \infty > 1.$$

By the ratio test, the series diverges.

SUMMARY OF TESTS SO FAR
We have been dealing with series $\sum a_k$ with positive terms.
(I) Usually, when a series has terms with powers, e.g.

$$\sum \frac{2^k}{1+3^k} \text{ or } \sum \frac{k^2}{2^k}$$

but no factorials, we use the root test.
(II) When there are also factorials, e.g.

$$\sum \frac{4^k}{k!},$$

we use the ratio test.
(III) When the terms have the form of a numerator and denominator with powers of k, e.g.,

$$\sum \frac{k^{3/2}+2}{4k^2+\sqrt{k}},$$

we use the limit comparison test (or comparison test) and compare to a p-series $\sum \frac{1}{k^p}$.

11.4 Absolute, Conditional Convergence

So far we have dealt mainly with series with positive terms. Now we study series $\sum a_k$ where different terms may have different signs. Very often, we can reduce this to studying $\sum |a_k|$.

Definition Absolute Convergence
If

$$\sum |a_k| \text{ converges,}$$

then we say that

$$\sum a_k \textbf{ converges absolutely.}$$

The resaon is that this is useful is:

Theorem 11.4.1 (page 657 of SHE) Absolute convergence implies convergence
If

$$\sum |a_k| \text{ converges,}$$

then

$$\sum a_k \text{ converges.}$$

Proof
Now

$$-|a_k| \leq a_k \leq |a_k|$$
$$\Rightarrow 0 \leq a_k + |a_k| \leq 2|a_k|.$$

By the comprison test, since $\sum |a_k|$ converges, also

$$\sum (a_k + |a_k|) \text{ converges.}$$

Then

$$\begin{aligned} \sum a_k &= \sum [(a_k + |a_k|) - |a_k|] \\ &= \sum (a_k + |a_k|) - \sum |a_k| \end{aligned}$$

is a difference of two convergent series, so $\sum a_k$ converges. ■

Example
Show that if $p > 1$,

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} \text{ converges.}$$

Solution
Here

$$a_k = \frac{(-1)^k}{k^p} \Rightarrow |a_k| = \frac{1}{k^p}.$$

As $p > 1$, we know that

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges.}$$

That is,

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} \text{ converges absolutely, and so converges.}$$

Remark
We shall soon see that for any $p > 0$,

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} \text{ converges,}$$

even though it does not converge absolutely for $p \leq 1$. We shall call this conditional convergence:

Definition Conditional Convergence
Suppose that

$$\sum a_k \text{ converges,}$$

but

$$\sum |a_k| \text{ diverges.}$$

Then we say that $\sum a_k$ **converges conditionally.**

Theorem 11.4.3 (Page 659 of SHE) Alternating Series Test
Let $a_k > 0$ for $k \geq 0$. Assume that
(i) $\{a_k\}$ is a decreasing sequence of positive numbers;
(ii)

$$\lim_{k \to \infty} a_k = 0.$$

Then

$$\sum_{k=0}^{\infty} (-1)^k a_k \text{ converges.}$$

Remark
We call series of this form **alternating series.**
Idea of Proof

The series converges because succesive terms cancel one another, due to opposite sign. There is a beautiful proof in SHE: Let $m \geq 1$, and consider s_{2m}, the even order partial sum. We see that

$$\begin{aligned} s_{2m} &= \sum_{k=0}^{2m} (-1)^k a_k \\ &= (a_0 - a_1) + (a_2 - a_3) + ... + (a_{2m-2} - a_{2m-1}) + a_{2m}. \end{aligned}$$

Here each term in brackets, namely

$$a_0 - a_1; a_2 - a_3; ...; a_{2m-2} - a_{2m-1}$$

is positive (recall $a_1 < a_0$ and $a_3 < a_2$ etc.). So

$$s_{2m} > 0 \text{ for } m \geq 1.$$

On the other hand,

$$\begin{aligned} s_{2m+2} &= s_{2m} + (-1)^{2m+1} a_{2m+1} + a_{2m+2} \\ &= s_{2m} - a_{2m+1} + a_{2m+2} < s_{2m}, \end{aligned}$$

since $a_{2m+1} > a_{2m+2}$. Thus $\{s_{2m}\}$ is a decreasing sequence of positive numbers. As it is bounded below (by 0), we have

$$\lim_{m\to\infty} s_{2m} = L$$

for some $L > 0$. Then also the odd order partial sums converge to the same limit:

$$\begin{aligned} s_{2m+1} &= s_{2m} + (-1)^{2m+1} a_{2m+1} \\ &= s_{2m} - a_{2m+1}. \end{aligned}$$

Hence

$$\begin{aligned} \lim_{m\to\infty} s_{2m+1} &= \lim_{m\to\infty} (s_{2m} - a_{2m+1}) \\ &= L - 0 = L. \end{aligned}$$

Hence

$$\lim_{m\to\infty} s_m = L.$$

■

Example
Let $p > 0$. Show that

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} \text{ converges.}$$

For which $p > 0$ does it converge absolutely? For which $p > 0$ does it converge conditionally?

Solution

Here

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$$

has the form

$$\sum_{k=1}^{\infty} (-1)^k a_k,$$

where

$$a_k = \frac{1}{k^p} \text{ is positive, decreasing (as } k \text{ increases)}$$

and

$$\lim_{k \to \infty} a_k = 0.$$

By the alternating series test,

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} = \sum_{k=1}^{\infty} (-1)^k a_k \text{ converges.}$$

We already saw that for $p > 1$, the series converges absolutely. If $p \leq 1$, then

$$\sum_{k=1}^{\infty} \left| (-1)^k a_k \right| = \sum_{k=1}^{\infty} \frac{1}{k^p} \text{ diverges,}$$

as it is a $p-$series, with $p \leq 1$. Thus

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} \text{ converges conditionally for } p \leq 1.$$

Summary

When you see a series with terms having alternating sign, you can try apply the alternating series test, **provided** you can easily check that the terms decrease in magnitude, and have limit 0. In many cases, however, it is easier to take absolute values, and test for absolute convergence. If there is absolute convergence, then there is convergence.