11. 5 Taylor Polynomials

(January 13, 2006)

Taylor polynomials are used to approximate functions. Let f be a function for which

$$f'(0), f''(0), f'''(0) \ldots$$

all exist. Then the constant function

$$P_0(x) = f(0)$$

has the same value as f at 0. The linear polynomial

$$P_1(x) = f(0) + f'(0)x$$

has the same value and first derivative as f at 0. That is,

$$P_1(0) = f(0); P_1'(0) = f'(0).$$

Similarly

$$P_2(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!}$$

has the same value and first two derivatives as f at 0 :

$$P_2(0) = f(0); P_2'(0) = f'(0); P_2''(0) = f''(0).$$

More generally:

Definition
The nth Taylor polynomial for f at 0 is

$$P_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \ldots + f^{(n)}(0)\frac{x^n}{n!}.$$

Note that

$$P_n(0) = f(0); P_n'(0) = f'(0); \ldots; P^{(n)}(0) = f^{(n)}(0).$$

Example 1
Let

$$f(x) = e^x.$$

Then

$$f'(x) = e^x; f''(x) = e^x; \ldots f^{(n)}(x) = e^x,$$

so

$$f'(0) = 1; f'(0) = 1; \ldots; f^{(n)}(0) = 1.$$

The nth Taylor polynomial for f is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}.$$

Example 2
Let

$$f(x) = \sin x.$$

Then

$$\begin{aligned} f'(x) &= \cos x; \\ f''(x &= -\sin x; \\ f'''(x) &= -\cos x; \\ f^{(4)}(x) &= \sin x; \\ &\vdots \end{aligned}$$

so

$$\begin{aligned} f(0) &= 0; \\ f'(0) &= 1; \\ f''(0) &= 0; \\ f'''(0) &= -1 \\ f^{(4)}(0) &= 0 \\ &\vdots \end{aligned}$$

Then

$$\begin{aligned} P_0(x) &= 0; \\ P_1(x) &= 0 + 1x = x \\ P_2(x) &= 0 + 1x + 0\frac{x^2}{2!} = x; \\ P_3(x) &= 0 + 1x + 0\frac{x^2}{2!} - 1\frac{x^3}{3!} = x - \frac{x^3}{3!}; \\ P_4(x) &= 0 + 1x + 0\frac{x^2}{2!} - 1\frac{x^3}{3!} + 0\frac{x^4}{4!} = x - \frac{x^3}{3!} \\ &\vdots \end{aligned}$$

It is important to know how close is $P_n(x)$ to $f(x)$:

Theorem 11.5.1 (Taylor's Theorem, page 667 of SHE, 9th edn.)
Suppose that f has $n+1$ continuous derivatives on an open interval I that contains 0. Then for $x \in I$,

$$f(x) = P_n(x) + R_n(x),$$

where

$$P_n(x) = f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2!} + \ldots + f^{(n)}(0)\,\frac{x^n}{n!}$$

and

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)\,(x-t)^n\,dt.$$

Proof for $n = 0$
We see that the given $R_0(x)$ for $n = 0$ is

$$\begin{aligned} R_0(x) &= \frac{1}{0!}\int_0^x f'(t)\,(x-t)^0\,dt \\ &= \int_0^x f'(t)\,dt \\ &= f(x) - f(0) \\ &= f(x) - P_0(x). \end{aligned}$$

Then

$$f(x) = P_0(x) + R_0(x). \tag{1}$$

Proof for $n = 1$
We integrate by parts in

$$\begin{aligned} R_0(x) &= \int_0^x f'(t)\,dt \\ &= \int_0^x u(t)\,dv(t) \end{aligned} \tag{2}$$

where

$$\begin{aligned} u(t) &= f'(t); \\ \text{so } du(t) &= f''(t)\,dt \\ \text{and } v(t) &= t - x \\ \text{so } dv(t) &= 1dt. \end{aligned}$$

(Recall x is fixed in the integral, we integrate with respect to t). Integrating by parts in (2) gives

$$\begin{aligned}
R_0(x) &= [uv]_{t=0}^{t=x} - \int_0^x v(t)\,du(t) \\
&= f'(x)\,0 - f'(0)(-x) - \int_0^x (t-x)\,f''(t)\,dt \\
&= f'(0)\,x + \frac{1}{1!}\int_0^x f''(t)(x-t)^1\,dt \\
&= f'(0)\,x + R_1(x),
\end{aligned}$$

with the notation above. Then substitute in (1):

$$\begin{aligned}
f(x) &= P_0(x) + R_0(x) \\
&= f(0) + f'(0)\,x + R_1(x) \\
&= P_1(x) + R_1(x).
\end{aligned}$$

Proof for $n = 2$
Integrate by parts in

$$\begin{aligned}
R_1(x) &= \frac{1}{1!}\int_0^x f''(t)(x-t)^1\,dt \\
&= \frac{1}{1!}\int_0^x u(t)\,dv(t)
\end{aligned}$$

where

$$\begin{aligned}
u(t) &= f''(t); \\
\text{so } du(t) &= f'''(t)\,dt \\
\text{and } v(t) &= \frac{(x-t)^2}{2!} \\
\text{so } dv(t) &= (x-t)\,dt.
\end{aligned}$$

You do this as an exercise.
Proof for general n
Do this by induction on n, each time integrating by parts in the previous remainder. ■

Corollary 11.5.2 (Lagrange Form of Remainder, page 667 of SHE, 9th edn.)
Suppose that f has $n+1$ continuous derivatives on an open interval I that contains 0. Let $x \in I$. Then

$$f(x) = P_n(x) + R_n(x),$$

where

$$P_n(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{2!} + ... + f^{(n)}(0)\frac{x^n}{n!}$$

and for some c between 0 and x,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.$$

Proof (uses Mean Value Theorem for Integrals). ■

Remarks

(a) We do not know where c is. But we do know that if J is an interval containing 0 and x, and if

$$\left| f^{(n+1)}(t) \right| \leq M \text{ for all } t \in J$$

then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x|^{n+1}.$$

However, M will usually depend on n, and we have to be careful to take account of this!

(b) We want

$$\lim_{n\to\infty} R_n(x) = 0$$

for then

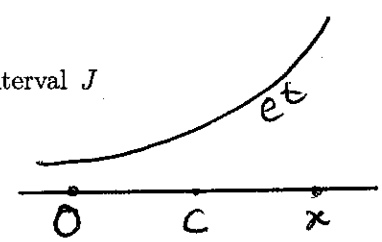
$$\lim_{n\to\infty} P_n(x) = f(x).$$

Example 1

Let $f(x) = e^x$ so that

$$P_n(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}.$$

By the remark above, if M is a bound on $f^{(n+1)}(t) = e^t$ in an interval J containing 0 and x, then



$$|R_n(x)| \leq \frac{M}{(n+1)!} |x|^{n+1}.$$

Now the big deal here is that $f^{(n+1)}(t) = e^t$ does not depend on n, so we can just take M to be the largest value of e^t in an interval containing 0 and x. Then as

$$\lim_{n\to\infty} \frac{1}{(n+1)!} |x|^{n+1} = 0,$$

so

$$\lim_{n\to\infty} R_n(x) = 0$$

and

$$\lim_{n\to\infty} P_n(x) = f(x),$$

that is

$$\lim_{n\to\infty}\left(1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!}\right)=e^x.$$

Example 2
Let

$$f(x)=\ln(1+x)\Rightarrow f(0)=0.$$

Here

$$\begin{aligned}
f'(x) &= \frac{1}{1+x}\Rightarrow f'(0)=1;\\
f''(x) &= \frac{-1}{(1+x)^2}\Rightarrow f''(0)=-1;\\
f'''(x) &= \frac{2}{(1+x)^3}\Rightarrow f'''(0)=2;\\
f^{(4)}(x) &= -\frac{3\cdot 2}{(1+x)^4}\Rightarrow f^{(4)}(0)=-3!\\
&\vdots\\
f^{(n)}(x) &= (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}\Rightarrow f^{(n)}(0)=(-1)^{n-1}(n-1)!.
\end{aligned}$$

Then

$$\begin{aligned}
P_n(x) &= f(0)+f'(0)x+f''(0)\frac{x^2}{2!}+...+f^{(n)}(0)\frac{x^n}{n!}\\
&= 0+x-\frac{x^2}{2!}+\frac{2x^3}{3!}-\frac{3!x^4}{4!}+...+(-1)^{n-1}\frac{(n-1)!}{n!}x^n\\
&= 0+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...+(-1)^{n-1}\frac{x^n}{n}.
\end{aligned}$$

We want to show that

$$\lim_{n\to\infty}R_n(x)=0$$

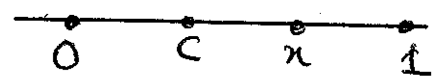
so that

$$\lim_{n\to\infty}P_n(x)=f(x).$$

We are going to do this for two ranges of x, separately for $x\in[0,1]$ and $x\in(-1,0)$.
Case I: $x\in[0,1]$

Here we use the (easier) Lagrange form of the remainder:

$$
\begin{aligned}
R_n(x) &= f^{(n+1)}(c)\frac{x^{n+1}}{(n+1)!} \\
&= \frac{(-1)^n}{(1+c)^{n+1}}n!\frac{x^{n+1}}{(n+1)!} \\
&= \frac{(-1)^n}{n+1}\left(\frac{x}{1+c}\right)^{n+1}.
\end{aligned}
$$



Here as $x > 0$ and c is between x and 0, also $c > 0$. Then

$$
\frac{x}{1+c} \leq x \leq 1,
$$

so

$$
\begin{aligned}
|R_n(x)| &= \frac{1}{n+1}\left(\frac{x}{1+c}\right)^{n+1} \\
&\leq \frac{1}{n+1} \to 0,
\end{aligned}
$$

as $n \to \infty$. Thus, as desired,

$$
\lim_{n\to\infty} R_n(x) = 0,
$$

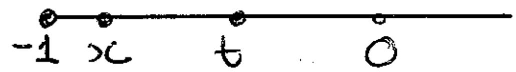
and hence

$$
\lim_{n\to\infty} P_n(x) = f(x).
$$

Case II: $x \in (-1, 0)$ Difficult!
Here the Lagrange form of the remainder does not work and we have to go use the more difficult integral form of the remainder. Now

$$
\begin{aligned}
R_n(x) &= \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt \\
&= \frac{1}{n!}\int_0^x (-1)^n\frac{n!}{(1+t)^{n+1}}(x-t)^n\,dt \\
&= (-1)^n\int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}dt. \qquad (1)
\end{aligned}
$$

Note that $t-x>0, 1+t>0$ for $t\in[x,0)$, so



$$\begin{aligned}
|R_n(x)| &= \left|(-1)^n\int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}dt\right| \\
&= \left|\int_0^x \left(\frac{x-t}{1+t}\right)^n \frac{1}{1+t}dt\right| \\
&\le \int_x^0 \left|\frac{x-t}{1+t}\right|^n \frac{1}{1+t}dt \\
&= \int_x^0 \left(\frac{t-x}{1+t}\right)^n \frac{1}{1+t}dt \\
&= \left(\frac{c_n-x}{1+x}\right)^{n+1} \frac{1}{1+c_n}(0-x),
\end{aligned}$$

for some c_n between x and 0, by the Mean Value Theorem for Integrals. Here $|x|<1$ and $x<c_n<0$, so

$$c_n < c_n|x|$$

$$\begin{aligned}
&\Rightarrow\ c_n+|x| < c_n|x|+|x| = (c_n+1)|x| \\
&\Rightarrow\ \frac{c_n+|x|}{c_n+1} < |x| \\
&\Rightarrow\ \frac{c_n-x}{c_n+1} < |x|.
\end{aligned}$$

Then substituting above, and as $1+c_n>1+x$,

$$|R_n(x)| < |x|^{n+1}\frac{1}{1+x} \to 0, n\to\infty.$$

Thus, as desired,

$$\lim_{n\to\infty} R_n(x) = 0,$$

and hence

$$\lim_{n\to\infty} P_n(x) = f(x).$$

Summary
For all $x\in(-1,1]$,

$$\lim_{n\to\infty} P_n(x) = f(x),$$

that is

$$\lim_{n\to\infty}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n-1}\frac{x^n}{n}\right) = \log(1+x).$$

Numerical Calculations

(Page 673 of SHE, 9th edn.)

Sometimes we want to know how large n must be to ensure that P_n approximates $f(x)$ with a given accuracy. For example, how large must n be for

$$|R_n(x)| = |f(x) - P_n(x)| \leq 0.01?$$

Or, for a given n, what is the largest error $|f(x) - P_n(x)|$, when x ranges over $[a, b]$?

Example

Determine the largest possible error when we use $P_6(x)$ to approximate e^x for $x \in [0,1]$.

Solution

From Lagrange's form for the remainder,

$$\begin{aligned} f(x) - P_6(x) &= R_6(x) \\ &= \frac{1}{7!} f^{(7)}(c)\, x^7 \\ &= \frac{e^c}{7!} x^7, \end{aligned}$$

for some c between 0 and x. Here as the function e^x is increasing, and $x \in [0,1]$

$$e^0 \leq e^c \leq e^x \leq e^1,$$

so

$$|R_6(x)| = \frac{e^c}{7!} x^7 \leq \frac{e}{7!}.$$

We can now use an estimate for e. One can use a calculator, or just use $e \leq 3$. Then

$$|R_6(x)| \leq \frac{3}{7!} = \frac{1}{1680} < 0.0006$$

Example

Estimate $e^{0.2}$ correct to three decimal places.. (This means with an error less than 0.0005, so that we do not round up the error to the 3rd decimal). Recall that

$$P_n(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}$$

and by the Lagrange form above, with $x = 0.2$,

$$\begin{aligned} |R_n(x)| &\leq \frac{1}{(n+1)!} f^{(n+1)}(c)\, x^{n+1} \\ &= \frac{1}{(n+1)!} e^c x^{n+1} \\ &\leq \frac{1}{(n+1)!} e^{0.2} x^{n+1}. \end{aligned}$$

For $e^{0.2}$, we just use the upper bound $e^1 \leq 3$, and then

$$|R_n(0.2)| \leq \frac{3}{(n+1)!}(0.2)^{n+1}.$$

We want

$$\frac{3}{(n+1)!}(0.2)^{n+1} < 0.0005. \tag{1}$$

We can just calculate the left-hand side until this becomes true:

n	$R_n(0.2)$	
1	$\frac{3}{2!}(0.2)^2$	0.06...
2	$\frac{3}{3!}(0.2)^3$	0.004...
3	$\frac{3}{4!}(0.2)^4$	0.0002...

So we can use $n = 3$ and

$$\begin{aligned} P_3(0.2) &= 1 + 0.2 + \frac{(0.2)^2}{2!} + \frac{(0.2)^3}{3!} \\ &= 1.22133... \end{aligned}$$

Remark
See SHE, page 673, for a slightly different way: one solves for the smallest n satisfying (1) using inequalities.

11.6 Taylor Polynomials in $x-a$

We can expand functions in Taylor series about points a other than 0. This involves the numbers $f(a), f'(a), f''(a) \dots$ instead of $f(0), f'(0), f''(0) \dots$.

Definition
The nth Taylor polynomial of g in powers of $x-a$ is

$$P_n(x) = g(a) + g'(a)(x-a) + g''(a)\frac{(x-a)^2}{2!} + \dots + g^{(n)}(a)\frac{(x-a)^n}{n!}$$

Theorem 11.6.1 Taylor's Theorem (p. 677 of SHE, 9th edn)
Let $a \in \mathbb{R}$. Suppose that g has $n+1$ continuous derivatives in an open interval I containing a. Then for $x \in I$,

$$g(x) = P_n(x) + R_n(x),$$

where P_n is as above and

$$R_n(x) = \frac{1}{n!}\int_a^x g^{(n+1)}(t)(x-t)^n\,dt.$$

Idea of Proof
Let

$$f(x) = g(x+a).$$

Then

$$\begin{aligned} f(0) &= g(a); \\ f'(0) &= g'(a); \\ f''(0) &= g''(a); \\ &\vdots \end{aligned}$$

So we can apply Taylor"s Theorem for f at 0, and then go back and rewrite everything in terms of g. ■

Example 1
Let

$$g(x) = x^3 \text{ and } a = 2.$$

Expand g in powers of $x-2$.
Solution
As $a = 2$,

$$P_n(x) = g(2) + g'(2)(x-2) + g''(2)\frac{(x-2)^2}{2!} + \dots + g^{(n)}(2)\frac{(x-2)^n}{n!}.$$

Here

$$\begin{aligned} g(x) &= x^3 \Rightarrow g(2) = 8; \\ g'(x) &= 3x^2 \Rightarrow g'(2) = 12; \\ g''(x) &= 6x \Rightarrow g''(2) = 12; \\ g'''(x) &= 6 \Rightarrow g'''(2) = 6; \\ g^{(n)}(x) &= 0 \text{ for } n \geq 4. \end{aligned}$$

Then for $n \geq 3$,

$$\begin{aligned} P_n(x) &= 8 + 12(x-2) + 12\frac{(x-2)^2}{2!} + 6\frac{(x-2)^3}{3!} \\ &= 8 + 12(x-2) + 6(x-2)^2 + (x-2)^3. \end{aligned}$$

In fact, one can use the remainder formula for $R_n(x)$ to show that

$$R_n(x) = \frac{1}{n!}\int_a^x g^{(n+1)}(t)(x-t)^n\,dt = 0 \text{ for } n \geq 3,$$

so

$$P_n(x) = g(x) \text{ for } n \geq 3,$$

that is,

$$8 + 12(x-2) + 6(x-2)^2 + (x-2)^3 = x^3.$$

Example 2
Expand $g(x) = x\ln x$ in powers of $x-1$, that is find its nth Taylor polynomial about $a = 1$.
Solution

$$\begin{aligned} g(x) &= x\ln x \Rightarrow g(1) = 0; \\ g'(x) &= 1\cdot\ln x + x\cdot\frac{1}{x} \\ &= \ln x + 1 \Rightarrow g'(1) = 1; \\ g''(x) &= \frac{1}{x} \Rightarrow g''(1) = 1; \\ g'''(x) &= -\frac{1}{x^2} \Rightarrow g'''(1) = -1; \\ g^{(4)}(x) &= \frac{2}{x^3} \Rightarrow g^{(4)}(1) = 2; \\ g^{(5)}(x) &= -\frac{3\cdot 2}{x^4} \Rightarrow g^{(5)}(1) = -3! \\ &\vdots \\ g^{(n)}(x) &= (-1)^n\frac{(n-2)!}{x^{n-1}} \Rightarrow g^{(n)}(1) = (-1)^n(n-2)!, n \geq 2. \end{aligned}$$

Then

$$\begin{aligned} P_n(x) &= g(1) + g'(1)(x-1) + g''(1)\frac{(x-1)^2}{2!} + \ldots + g^{(n)}(1)\frac{(x-1)^n}{n!} \\ &= 0 + 1(x-1) + 1\frac{(x-1)^2}{2!} - 1\frac{(x-1)^3}{3!} + \ldots + (-1)^n (n-2)!\frac{(x-1)^n}{n!} \\ &= (x-1) + \frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!} + \ldots + (-1)^n \frac{(x-1)^n}{n(n-1)}. \end{aligned}$$