1502 B6

Practice Mid Term I

Time: 1hrs 20 min.

Write your name in the top left corner. Attempt all questions. You must show all work in order to obtain credit

1. Find the Taylor polynomial P_4 for the function $f(x) = \sqrt{1+x}$. (4pts)

$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \qquad f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$$

$$f'''(x) = +\frac{1}{4}\cdot\frac{3}{2}(1+x)^{-\frac{5}{2}} \qquad f^{(4)}(x) = -\frac{1}{4}\cdot\frac{3}{2}\cdot\frac{5}{2}(1+x)^{-\frac{7}{2}}$$

$$P_4(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$$

2. Write down a non-constant function f such that $f(x) = P_{100}(x)$, where P_{100} is the Taylor polynomial of f of degree 100. (2pts)

Any polynomial of degree ≤ 100 will do.

For example, $f(x) = x$

3. Express $g(x) = x^n$ in powers of $(x-1)$. (4pts)

$$g^{(k)}(x) = n(n-1)\cdots(n-k+1)x^{n-k}$$

$$g(x) = \sum_{k\geq 0} \frac{g^{(k)}(1)}{k!}(x-1)^k$$

$$= \sum_{k\geq 0} \frac{n(n-1)\cdots(n-k+1)}{k!}(x-1)^k$$

$$= \sum_{k\geq 0} \binom{n}{k}(x-1)^k = \sum_{k=0}^{n} \binom{n}{k}(x-1)^k$$

4. Evaluate the following limits (or prove that the limits do not exist). (6pts)

(a)

$$\lim_{x \to 0} \frac{x}{\sin x}$$

By L'Hopital's rule.

$$\lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \frac{1}{\cos x} = 1$$

(b)

$$\lim_{x \to 0} x \sin 1/x$$

Since $-1 \leq \sin \frac{1}{x} \leq 1$ for all $x \neq 0$

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0$$

(c)

$$\lim_{x \to 0} \frac{1}{x} \sin 1/x$$

As $x \to 0$ $\frac{\sin \frac{1}{x}}{x}$ takes every value between $\pm \infty$ and so the limit DOES NOT EXIST.

5. Evaluate the improper integral (4pts)

$$\int_0^1 \frac{dx}{\sqrt{x}}.$$

$$\frac{1}{\sqrt{x}} \longrightarrow \infty \quad \text{as} \quad x \longrightarrow 0+ .$$

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b \to 0+} \int_b^1 x^{-\frac{1}{2}}\, dx$$

$$= \lim_{b \to 0^+} \left| \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right|_b^1 = \lim_{b \to 0^+} \left| 2x^{\frac{1}{2}} \right|_b^1$$

$$= \lim_{b \to 0^+} \left[2 - 2b^{\frac{1}{2}} \right] = 2$$

6. Consider the infinite series,

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)}.$$

Express the partial sums s_n directly as a function of n and evaluate the limit $\lim_{n\to\infty} s_n$. (3pts)

$$S_n = \sum_{k=0}^{n} \frac{1}{(2k+1)(2k+3)}$$

$$= \sum_{k=0}^{n} \frac{1}{2} \left[\frac{1}{2k+1} - \frac{1}{2k+3} \right]$$

$$= \frac{1}{2} \left(\sum_{k=0}^{n} \frac{1}{2k+1} - \sum_{k=0}^{n} \frac{1}{2k+3} \right)$$

$$= \frac{1}{2} \left(\frac{1}{1} - \frac{1}{2n+3} \right) = \frac{1}{2} \left(1 - \frac{1}{2n+3} \right)$$

$$\lim_{n \to \infty} S_n = \frac{1}{2}.$$

7. Prove the convergence or divergence of the following series. (3pts)

$$\sum_{k=0}^{\infty} \frac{k^9 + k^5 + 17k + 1}{k^{10} + 113}.$$

Justify your carefully in order to obtain any credit.

Compare with the harmonic series

$$\sum_{K=1}^{\infty} \frac{1}{K}.$$

By the Limit Comparison Test

$$\lim_{k \to \infty} \frac{1}{K} \cdot \frac{K^{10} + 113}{k^9 + k^5 + 17K + 1} = 1.$$

Hence, given serise DIVERGES because the harmonic series does so.

8. State carefully the ratio test for convergence of a series with positive terms. (4pts)

9. What can you deduce from the ratio test regarding convergence or divergence of the harmonic series, (2pts)

$$\sum_{k=1}^{\infty} \frac{1}{k}?$$

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} \frac{k}{k+1} = 1$$

So the RATIO TEST is <u>INCONCLUSIVE</u>

for the harmonic series.

10. Let $\{a_k\}$ be a decreasing sequence of positive numbers. Consider the alternating series

$$\sum_{k=0}^{\infty} (-1)^k a_k.$$

Prove that the odd partial sums s_{2m+1} are all positive and form an increasing sequence. (3pts)

$$s_{2m+1} = a_0 - a_1 + a_2 \cdots + a_{2m} - a_{2m+1}$$

$$= (a_0 - a_1) + (a_2 - a_3) + \cdots + (a_{2m} - a_{2m+1})$$

Since $a_0 > a_1 > a_2 \cdots\cdots$,

each term in parenthesis is > 0.

This shows $s_{2m+1} > 0$.

$$s_{2m+3} = s_{2m+1} + (a_{2m+2} - a_{2m+3})$$

Since $a_{2m+2} - a_{2m+3} > 0$, we have

$s_{2m+3} > s_{2m+1}$ and hence they form an increasing sequence.

11. Test the following series for absolute as well as conditional convergence. (3pts)

$$\sum_{k=0}^{\infty}(-1)^k \frac{k}{k^2+1}$$

$\left\{\frac{K}{K^2+1}\right\}$ is a decreasing sequence for all sufficiently large K.
So the series converges by the alternating series test.
But, $\sum_{K=0}^{\infty} \frac{K}{K^2+1}$ is divergent. (Compare with the harmonic series). So the given series is CONDITIONALLY CONVERGENT.

12. Find the interval of convergence of the following power series. (2pts)

$$\sum_{k=0}^{\infty} \frac{x^k}{k^k}.$$

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CONSIDER THE SERIES

$\sum_{K=1}^{\infty} \frac{|x|^K}{K^K}$. Apply RATIO TEST.

$$\lim_{K\to\infty} \frac{|x|^{K+1}}{(K+1)^{K+1}} \cdot \frac{K^K}{|x|^K} = \lim_{K\to\infty} \frac{K^K}{(K+1)^{K+1}} \cdot |x|$$

$$= \lim_{K\to\infty} \left(\frac{K}{K+1}\right)^K \cdot \frac{1}{K+1} \cdot |x| = 0$$

Interval of convergence is $(-\infty, \infty)$.