ON HOMOTOPY TYPES OF LIMITS OF SEMI-ALGEBRAIC SETS AND ADDITIVE COMPLEXITY OF POLYNOMIALS

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Abstract. We prove that the number of homotopy types of limits of one-parameter semi-algebraic families of closed and bounded semi-algebraic sets is bounded singly exponentially in the additive complexity of any quantifier-free first order formula defining the family. As an important consequence, we derive that the number of homotopy types of semi-algebraic subsets of $\mathbb{R}^k$ defined by a quantifier-free first order formula Φ, where the sum of the additive complexities of the polynomials appearing in Φ is at most a, is bounded by $2^{(ka)^{O(1)}}$. This proves a conjecture made in [5].

1. Introduction and statement of the main results

If S is a semi-algebraic subset of $\mathbb{R}^k$ defined by a quantifier-free first order formula Φ, then various topological invariants of S (such as the Betti numbers) can be bounded in terms of the "format" of the formula Φ (we define format of a formula more precisely below). The first results in this direction were proved by Oleĭnik and Petrovskiĭ [13, 14] (also independently by Thom [15], and Milnor [12]) who proved singly exponential bounds on the Betti numbers of real algebraic varieties in $\mathbb{R}^k$ defined by polynomials of degree bounded by d. These results were extended to more general semi-algebraic sets in [1, 10]. As a consequence of more general finiteness results of Pfaffian functions, Khovanskiĭ [11] proved singly exponential bounds on the number of connected components of real algebraic varieties defined by polynomials with a fixed number of monomials. We refer the reader to the survey article [3] for a more detailed survey of results on bounding the Betti numbers of semi-algebraic sets.

A second type of quantitative results on the topology of semi-algebraic sets, more directly relevant to the current paper, seek to obtain tight bounds on the number of different topological types of semi-algebraic sets definable by first order formulas of bounded format. It follows from the well-known Hardt's triviality theorem for o-minimal structures (see [16, 9]) that this number is finite for the two different notions of format discussed in the previous paragraph. However, the quantitative bounds that follow from the proof of Hardt's theorem give only doubly exponential bounds on the the number of topological types (unlike the singly exponential bounds on the Betti numbers). Tighter (i.e. singly exponential) bounds have been obtained on the number of possible *homotopy types* of semi-algebraic sets defined by different classes of formulas of bounded format [5, 2]. The main motivation behind this paper is to obtain a singly exponential bound on the number of distinct homotopy types

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of semi-algebraic sets defined by polynomials of bounded "additive complexity" (defined below) answering a question posed in [5].

Additive complexity is a measure of complexity of real polynomials introduced into real algebraic geometry by Benedetti and Risler in the book [7]. Roughly speaking the additive complexity of a polynomial (see Definition 1.6 below for a precise definition) is bounded from above by the number of additions in any straight line program (allowing divisions) that computes the values of the polynomial at generic points of $\mathbb{R}^n$. A surprising fact conjectured in [7], and proved by Coste [8] and van den Dries [16], is that the number of topological types of real algebraic varieties defined by polynomials of bounded additive complexity is finite.

In [5] a much more restricted notion of complexity of a polynomial was introduced, whose definition is similar to that of additive complexity except that no divisions were allowed in the straight line program. We will refer to this notion as "division-free additive complexity" in this paper.[1] Note that the additive complexity of a polynomial is clearly at most its division-free additive complexity, but can be much smaller (see Example 1.7 below). Notice also that the additive complexity (as well as the division-free additive complexity) of a polynomial $P \in \mathbb{R}[X_1, \ldots, X_k]$ is at most the number of monomials appearing in the support of P. Hence, quantitative results about the topology of semi-algebraic sets (such as singly exponential bounds on the Betti numbers, homotopy types etc.) in terms of additive complexity are often stronger than the corresponding statements about fewnomials.

1.1. **Bounding the number of homotopy types of semi-algebraic sets.** The problem of obtaining tight quantitative bounds on the topological types of semi-algebraic sets defined by formulas of bounded format was considered in [5]. Several results (with different notions of formats of formulas) were proved in [5], each giving an explicit singly exponential (in the number of variables and size of the format) bound on the number of homotopy types of semi-algebraic subsets of $\mathbb{R}^k$ defined by formulas having format of bounded size. However, the case of additive complexity was left open in [5], and only a strictly weaker result was proved in the case of division-free additive complexity. In order to state this result precisely, we need a few preliminary definitions.

Definition 1.1 ([5])**.** A polynomial $P \in \mathbb{R}[X_1, \ldots, X_k]$ has *division-free additive complexity at most a* if there are polynomials $Q_1, \ldots, Q_a \in \mathbb{R}[X_1, \ldots, X_k]$ such that

(i) $Q_1 = u_1 X_1^{\alpha_{11}} \cdots X_k^{\alpha_{1k}} + v_1 X_1^{\beta_{11}} \cdots X_k^{\beta_{1m}}$,
where $u_1, v_1 \in \mathbb{R}$, and $\alpha_{11}, \ldots, \alpha_{1k}, \beta_{11}, \ldots, \beta_{1k} \in \mathbb{N}$;

(ii) $Q_j = u_j X_1^{\alpha_{j1}} \cdots X_k^{\alpha_{jk}} \prod_{1 \le i \le j-1} Q_i^{\gamma_{ji}} + v_j X_1^{\beta_{j1}} \cdots X_k^{\beta_{jk}} \prod_{1 \le i \le j-1} Q_i^{\delta_{ji}}$,
where $1 < j \le a$, $u_j, v_j \in \mathbb{R}$, and $\alpha_{j1}, \ldots, \alpha_{jk}, \beta_{j1}, \ldots, \beta_{jk}, \gamma_{ji}, \delta_{ji} \in \mathbb{N}$ for $1 \le i < j$;

(iii) $P = c X_1^{\zeta_1} \cdots X_k^{\zeta_k} \prod_{1 \le j \le a} Q_j^{\eta_j}$,
where $c \in \mathbb{R}$, and $\zeta_1, \ldots, \zeta_k, \eta_1, \ldots, \eta_a \in \mathbb{N}$.

In this case we say that the above sequence of equations is a division-free additive representation of P of length a.

In other words, P has division-free additive complexity at most a if there exists a straight line program which, starting with variables $X_1, \ldots, X_m$ and constants in

[1] Note that what we call "additive complexity" is called "rational additive complexity" in [5], and what we call "division-free additive complexity" is called "additive complexity" there.

$\mathbb{R}$ and applying additions and multiplications, computes P and which uses at most a additions (there is no bound on the number of multiplications).

Example 1.2. The polynomial $P := (X+1)^d \in \mathbb{R}[X]$ with $0 < d \in \mathbb{Z}$, has $d+1$ monomials when expanded but division-free additive complexity at most 1.

Notation 1.3. We denote by $\mathcal{A}_{k,a}^{\mathrm{div-free}}$ the family of ordered (finite) lists $\mathcal{P} = (P_1, \ldots, P_s)$ of polynomials $P_i \in \mathbb{R}[X_1, \ldots, X_k]$, with the division-free additive complexity of every P_i not exceeding a_i, with $a = \sum_{1 \le i \le s} a_i$.

Suppose that ϕ is a Boolean formula with atoms $\{p_i, q_i, r_i \mid 1 \le i \le s\}$. For an ordered list $\mathcal{P} = (P_1, \ldots, P_s)$ of polynomials $P_i \in \mathbb{R}[X_1, \ldots, X_k]$, we denote by $\phi_{\mathcal{P}}$ the formula obtained from ϕ by replacing for each i, $1 \le i \le s$, the atom p_i (respectively, q_i and r_i) by $P_i = 0$ (respectively, by $P_i > 0$ and by $P_i < 0$).

Definition 1.4. We say that two ordered lists $\mathcal{P} = (P_1, \ldots, P_s)$, $\mathcal{Q} = (Q_1, \ldots, Q_s)$ of polynomials $P_i, Q_i \in \mathbb{R}[X_1, \ldots, X_k]$ have the same *homotopy type* if for any Boolean formula ϕ, the semi-algebraic sets defined by $\phi_{\mathcal{P}}$ and $\phi_{\mathcal{Q}}$ are homotopy equivalent.

The following theorem is proved in [5].

Theorem 1.5. [5] *The number of different homotopy types of ordered lists in* $\mathcal{A}_{k,a}^{\mathrm{div-free}}$ *does not exceed*

$$2^{O((k+a)a)^4}. \tag{1.1}$$

In particular, the number of different homotopy types of semi-algebraic sets defined by a fixed formula $\phi_{\mathcal{P}}$, where $\mathcal{P}$ varies over $\mathcal{A}_{k,a}^{\mathrm{div-free}}$, does not exceed (1.1).

The additive complexity of a polynomial is defined in [7] as follows.

Definition 1.6. A polynomial $P \in \mathbb{R}[X_1, \ldots, X_k]$ is said to have *additive complexity* at most a if there are *rational functions* $Q_1, \ldots, Q_a \in \mathbb{R}(X_1, \ldots, X_k)$ satisfying conditions (i), (ii), and (iii) in Definition 1.1 with $\mathbb{N}$ replaced by $\mathbb{Z}$. In this case we say that the above sequence of equations is an additive representation of P of length a.

Example 1.7. The polynomial $X^d + \cdots + X + 1 = (X^{d+1} - 1)/(X - 1) \in \mathbb{R}[X]$ with $0 < d \in \mathbb{Z}$, has additive complexity (but not division-free additive complexity) at most 2 (independent of d).

As before

Notation 1.8. We denote by $\mathcal{A}_{k,a}$ the family of ordered (finite) lists $\mathcal{P} = (P_1, \ldots, P_s)$ of polynomials $P_i \in \mathbb{R}[X_1, \ldots, X_k]$, with the additive complexity of every P_i not exceeding a_i, with $a = \sum_{1 \le i \le s} a_i$.

Note that Theorem 1.5 does not extend to the case of additive complexity and indeed it is conjectured in [5] (see also [7, Section 4.6.5]) that the number of different homotopy types of lists in $\mathcal{A}_{k,a}$ does not exceed

$$2^{(ka)^{O(1)}}.$$

In this paper we prove this conjecture. More formally

Theorem 1.9. *The number of different homotopy types of ordered lists in $\mathcal{A}_{k,a}$ does not exceed $2^{(ka)^{O(1)}}$.*

1.2. **Additive complexity and limits of semi-algebraic sets.** The proof of Theorem 1.5 in [5] proceeds by reducing the problem to the case of bounding the number of homotopy types of semi-algebraic sets defined by polynomials having a bounded number of monomials. The reduction is as follows. Let $\mathcal{P} \in \mathcal{A}_{k,a}^{\mathrm{div-free}}$ be an ordered list. For each polynomial $P_i \in \mathcal{P}$, $1 \leq i \leq s$, consider the sequence of polynomials $Q_{i1}, \ldots, Q_{ia_i}$ as in Definition 1.1, so that

$$P_i := c_i X_1^{\zeta_{i1}} \cdots X_k^{\zeta_{ik}} \prod_{1 \leq j \leq a_i} Q_{ij}^{\eta_{ij}}.$$

Introduce a_i new variables $Y_{i1}, \ldots, Y_{ia_i}$. Fix a semi-algebraic set $S \subset \mathbb{R}^m$, defined by a formula $\phi_{\mathcal{P}}$. Consider the semi-algebraic set $\widehat{S}$, defined by the conjunction of a 3-nomial equations obtained from equalities in (i), (ii) of Definition 1.1 by replacing Q_{ij} by Y_{ij} for all $1 \leq i \leq s$, $1 \leq j \leq a_k$, and the formula $\phi_{\mathcal{P}}$ in which every occurrence of an atomic formula of the kind $P_k * 0$, where $* \in \{=, >, <\}$, is replaced by the formula

$$c_i X_1^{\zeta_{i1}} \cdots X_k^{\zeta_{ik}} \prod_{1 \leq j \leq a_i} Y_{ij}^{\eta_{ij}} * 0.$$

Note that $\widehat{S}$ is a semi-algebraic set in $\mathbb{R}^{k+a}$.

Let $\rho : \mathbb{R}^{k+a} \to \mathbb{R}^k$ be the projection map on the subspace of coordinates $X_1, \ldots, X_k$. It is clear that the restriction $\rho_{\widehat{S}} : \widehat{S} \to S$ is a homeomorphism, and moreover $\widehat{S}$ is defined by polynomials having at most $k + a$ monomials.

Notice that for the map $\rho_{\widehat{S}}$ to be a homeomorphism it is crucial that the exponents $\eta_{ij}, \gamma_{ij}, \delta_{ij}$ be non-negative, and this restricts the proof to the case of division-free additive complexity. We overcome this difficulty as follows.

Given a polynomial $F \in \mathbb{R}[X_1, \ldots, X_k]$ with additive complexity bounded by a, we prove that F can be expressed as a quotient $\frac{P}{Q}$ with $P, Q \in \mathbb{R}[X_1, \ldots, X_k]$ with each P, Q having *division-free* additive complexity bounded by a (see Lemma 3.1 below). We then express the set of real zeros of F in $\mathbb{R}^k$ inside any fixed closed ball as the Hausdorff limit of a one-parameter semi-algebraic family defined using the polynomials P and Q (see Proposition 3.5 and the accompanying Example 3.6 below).

While the limits of one-parameter semi-algebraic families defined by polynomials with bounded division-free additive complexities themselves can have complicated descriptions which cannot be described by polynomials of bounded division-free additive complexity, the topological complexity (for example, measured by their Betti numbers) of such limit sets are well controlled. Indeed, the problem of bounding the Betti numbers of Hausdorff limits of one-parameter families of semi-algebraic sets was considered by Zell in [19], who proved a singly exponential bound on the Betti numbers of such sets. We prove in this paper (see Theorems 2.1 and 1.12 below) that the number of homotopy types of such limits can indeed be bounded single exponentially in terms of the format of the formulas defining the one-parameter family. The techniques introduced by Zell in [19] (as well certain semi-algebraic constructions described in [6]) play a crucial role in the proof of our bound. These intermediate results may be of independent interest.

Finally, applying Theorem 2.1 to the one-parameter family referred to in the previous paragraph, we obtain a bound on the number of homotopy types of real algebraic varieties defined by polynomials having bounded additive complexity. The semi-algebraic case requires certain additional techniques and is dealt with in Section 3.3.

1.3. **Homotopy types of limits of semi-algebraic sets.** In order to state our results on bounding the number of homotopy types of limits of one-parameter families of semi-algebraic sets we need to introduce some notation.

Notation 1.10 (Format of first-order formulas)**.** Suppose Φ is a quantifier-free first order formula defining a semi-algebraic subset of $\mathbb{R}^k$ involving s polynomials of degree at most d. In this case say that Φ has *dense format* (s, d, k). If the sum of the (division-free) additive complexities of the polynomials appearing in Φ does not exceed a, then we say that Φ has *(division-free) additive format* bounded by (a, k).

Notation 1.11. For $1 \leq p \leq q \leq k$, we denote by $\pi_{[p,q]} : \mathbb{R}^k = \mathbb{R}^{[1,k]} \to \mathbb{R}^{[p,q]}$ the projection

$$(x_1, \dots, x_k) \mapsto (x_p, \dots, x_q).$$

In case $p = q$ we will denote by π_p the projection $\pi_{[p,p]}$. For any semi-algebraic subset $S \subset \mathbb{R}^k$, and $X \subset \mathbb{R}^{[q,k]}$, denote by S_X the semi-algebraic set $\pi_{[1,q-1]}(\pi^{-1}_{[q,k]}(X) \cap S)$, and S_x rather than $S_{\{x\}}$. We also let $S_{<a}$ and $S_{\leq a}$ denote $S_{(-\infty,a)}$ and $S_{(-\infty,a]}$ for $a \in \mathbb{R}$.

We have the following theorem which might be of independent interest.

Theorem 1.12. *Fix $a, k \in \mathbb{N}$. There exists a collection $\mathcal{S}_{a,k}$ of semi-algebraic subsets of $\mathbb{R}^k$ such that $\#\mathcal{S}_{a,k} = 2^{(ka)^{O(1)}}$ and with the following property. If $T \subset \mathbb{R}^{k+1} \cap \{(\boldsymbol{x}, t) \in \mathbb{R}^{k+1} \mid t > 0\}$ is a bounded semi-algebraic set such that T_t is closed for each $t > 0$, T is described by a formula having additive format bounded by $(a, k+1)$, and $T_0 := \pi_{[1,k]}(\overline{T} \cap \pi^{-1}_{k+1}(0))$, then T_0 is homotopy equivalent to some $S \in \mathcal{S}_{a,k}$.*

The rest of the paper is devoted to the proofs of Theorems 1.12 and 1.9 and is organized as follows. We first prove a weak version (Theorem 2.1) of Theorem 1.12 in Section 2, in which the term "additive complexity" in the statement of Theorem 1.12 is replaced by the term "division-free additive complexity". Theorem 2.1 is then used in Section 3 to prove Theorem 1.9 after introducing some additional techniques, which in turn is used to prove Theorem 1.12.

2. Proof of a weak version of Theorem 1.12

In this section we prove the following weak version of Theorem 1.12 (using *division-free* additive format rather than additive format) which is needed in the proof of Theorem 1.9.

Theorem 2.1. *Fix $a, k \in \mathbb{N}$. There exists a collection $\mathcal{S}_{a,k}$ of semi-algebraic subsets of of $\mathbb{R}^k$ such that $\#\mathcal{S}_{a,k} = 2^{(ka)^{O(1)}}$ and with the following property. If $T \subset \mathbb{R}^{k+1} \cap \{(\boldsymbol{x}, t) \in \mathbb{R}^{k+1} \mid t > 0\}$ is a bounded semi-algebraic set such that T_t is closed for each $t > 0$, T is described by a formula having* division-free *additive format bounded by $(a, k+1)$, and $T_0 := \pi_{[1,k]}(\overline{T} \cap \pi^{-1}_{k+1}(0))$, then T_0 is homotopy equivalent to some $S \in \mathcal{S}_{a,k}$.*

2.1. **Outline of the proofs.** The main steps of the proofs are as follows. Let $T \subset \mathbb{R}^k \times \mathbb{R}_{>0}$, such that each fiber T_t is closed and bounded, and let T_0 be as in Theorem 2.1.

We first prove that for all small enough $\lambda > 0$, there exists a semi-algebraic surjection $f_\lambda : T_\lambda \to T_0$ which is metrically close to the identity map 1_{T_λ} (see Proposition 2.23 below). Using a semi-algebraic realization of the fibered join described in [6], we then consider for any fixed $p \geq 0$, a semi-algebraic set $\mathcal{J}^p_{f_\lambda}(T_\lambda)$ which is p-equivalent to T_0 (see Proposition 2.12). The definition of $\mathcal{J}^p_{f_\lambda}(T_\lambda)$ still involves the map f_λ, whose definition is not simple, and hence we cannot control the topological type of $\mathcal{J}^p_{f_\lambda}(T_\lambda)$ directly. However, the fact that f_λ is close to the identity map enables us to adapt the main technique in [19] due to Zell. We replace $\mathcal{J}^p_{f_\lambda}(T_\lambda)$ by another semi-algebraic set, which we denote by $\mathcal{D}^p_\varepsilon(T)$ (for $\varepsilon > 0$ small enough), which is homotopy equivalent to $\mathcal{J}^p_{f_\lambda}(T_\lambda)$, but whose definition no longer involves the map f_λ (Definition 2.17). It is now possible to bound the format of $\mathcal{D}^p_\varepsilon(T)$ in terms of the format of the formula defining T, which leads directly to proof of the following proposition.

Recall that

Definition 2.2 (p-equivalence)**.** A map $f : A \to B$ between two topological spaces is called a *p-equivalence* if the induced homomorphism between the homotopy groups,

$$f_* : \pi_i(A) \to \pi_i(B)$$

is an isomorphism for all $0 \leq i < p$, and an epimorphism for $i = p$, and we say that A is *p-equivalent to B.*

Proposition 2.3. *Let $T \subset \mathbb{R}^{k+1} \cap \{(\boldsymbol{x}, t) \in \mathbb{R}^{k+1} \mid t > 0\}$ be a bounded semi-algebraic set such that T_t is closed for each $t > 0$, and $p \geq 0$. Suppose that T is described by a formula having (division-free) additive (resp. dense) format bounded by $(a, k+1)$ (resp. $(s, d, k+1)$). Then, there exists a semi-algebraic set $\mathcal{D}^p \subset \mathbb{R}^N$ such that $\mathcal{D}^p$ is p-equivalent to $T_0 := \pi_{[1,k]}(\overline{T} \cap \pi^{-1}_{k+1}(0))$, and such that $\mathcal{D}^p$ is described by a formula having (division-free) additive (resp. dense) format bounded by (M, N) (resp. $(M', d+1, N)$) where $M = (p+1)(k+a+3) + 2\binom{p+1}{2}(k+2)$ and $N = (p+1)(k+1) + \binom{p+1}{2}$ (resp. $M' = (p+1)(s+2) + 3\binom{p+1}{2} + 3$).*

Finally, Theorem 2.1 is an easy consequence of Proposition 2.3.

2.2. **Preliminaries.** We need a few facts from the homotopy theory of finite CW-complexes.

We first prove a basic result about p-equivalences (defined earlier). It is clear that p-equivalence is not an equivalence relation (e.g. for any $p \geq 0$, the map taking $\mathbf{S}^p$ to a point is a p-equivalence, but no map from a point into $\mathbf{S}^p$ is one). However, we have the following.

Proposition 2.4. *Let A, B, C be finite CW-complexes with $\dim(A), \dim(B) \leq k$ and suppose that C is p-equivalent to A and B for some $p > k$. Then, A and B are homotopy equivalent.*

The proof of Proposition 2.4 will rely on the following well-known lemmas (see for example [17, pp. 70] or [18, Theorem 7.16]).

Lemma 2.5. *Let X, Y be CW-complexes and $f : X \to Y$ a p-equivalence. Then, for each CW-complex M, $\dim(M) \leq p$, the map*

$$\pi(\mathrm{Id}, f) : \pi(M, X) \to \pi(M, Y)$$

is surjective.

Lemma 2.6. *If A and B are finite CW-complexes, with $dim(A) < p$ and $dim(B) \leq p$, then every p-equivalence from A to B is a homotopy equivalence.*

Proof of Proposition 2.4. Suppose $f : C \to A$ and $g : C \to B$ are two p-equivalences. Applying Lemma 2.5 with $X = C$, $M = Y = A$, we have the map Id_B has a preimage under $\pi(\mathrm{Id}_B, f)$. Denote this preimage by $h : A \to C$. Then,

$$(\mathrm{Id}_A)_* = f_* \circ h_* : \pi_i(A) \to \pi_i(A),$$

is an isomorphism for every $i \geq 0$. In particular, since f is a p-equivalence, this implies that $h_* : \pi_i(A) \to \pi_i(C)$ is an isomorphism for $0 \leq i < p$. Composing h with g, and noting that g is also a p-equivalence we get that the map $(g \circ h)_* : \pi_i(A) \to \pi_i(B)$ is an isomorphism for all $i \geq 0$, and hence a weak equivalence. Since the spaces A and B are assumed to be finite CW-complexes, every weak equivalence is in fact a homotopy equivalence. □

We introduce some more notation.

Notation 2.7. For any $R \in \mathbb{R}$ with $R > 0$, we denote by $B_k(0, R) \subset \mathbb{R}^k$, the open ball of radius R centered at the origin.

Notation 2.8. For $P \in \mathbb{R}[X_1, \ldots, X_k]$ we denote by $\mathrm{Zer}(P, \mathbb{R}^k)$ the real algebraic set defined by $P = 0$.

Notation 2.9. For any first order formula Φ in the language of ordered fields with k free variables, we denote by $\mathrm{Reali}(\Phi)$ the semi-algebraic subset of $\mathbb{R}^k$ defined by Φ. Additionally, if $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k]$ consists of the polynomials appearing in Φ, then we call Φ a $\mathcal{P}$-formula.

A very important construction that we use later in the paper is an efficient semi-algebraic realization (up to homotopy) of the iterated fibered join of a semi-algebraic set over a semi-algebraic map. This construction was introduced in [6].

We use the lower case bold-face notation $\mathbf{x}$ to denote a point $\mathbf{x} = (x_1, \ldots, x_k)$ of $\mathbb{R}^k$, and upper-case $\mathbf{X} = (X_1, \ldots, X_k)$ to denote a *block of variables*.

Definition 2.10 (The semi-algebraic fibered join [6])**.** For a semi-algebraic subset $S \subset \mathbb{R}^k$ contained in $B_k(0, R)$, defined by a $\mathcal{P}$-formula Φ and $f : S \to T$ a semi-algebraic map, we define

$$\begin{aligned}\mathcal{J}_f^p(S) = &\{(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}, \mathbf{a}) \in \mathbb{R}^{(p+1)(k+1)+\binom{p+1}{2}} \mid \\ &\Omega^R(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}) \wedge \Theta_1(\mathbf{t}, \mathbf{a}) \wedge \Theta_2^\Phi(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}) \wedge \Theta_3(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}, \mathbf{a})\}\end{aligned}$$

where

$$
\begin{aligned}
\Omega^R &:= \bigwedge_{i=0}^{p} (|\mathbf{X}^i|^2 \leq R^2) \wedge |\mathrm{T}|^2 \leq 4, \\
\Theta_1 &:= \sum_{i=0}^{p} T_i = 1 \wedge \sum_{1 \leq i < j \leq p} A_{ij}^2 = 0, \\
\Theta_2^{\Phi} &:= \bigwedge_{i=0}^{p} (T_i = 0 \vee \Phi(\mathbf{X}^i)), \\
\Theta_3^{f} &:= \bigwedge_{0 \leq i < j \leq p} (T_i = 0 \vee T_j = 0 \vee |f(\mathbf{X}^i) - f(\mathbf{X}^j)|^2 = A_{ij}).
\end{aligned}
\tag{2.1}
$$

We denote the formula $\Omega^R \wedge \Theta_1 \wedge \Theta_2^{\Phi} \wedge \Theta_3^f$ by $\mathcal{J}_f^p(\Phi)$.

Remark 2.11. Definition 2.10 differs slightly from the original definition in [6] in that we have replaced the predicate $\wedge_{0 \leq i \leq p} T_i \geq 0$ in the original definition by the predicate $\sum_{0 \leq i \leq p} T_i^2 \leq 4$. Also, because of the application in [6] the semi-algebraic fibered join was defined as a semi-algebraic subset of a sphere of large enough radius. This is not essential in this paper and this leads to a further simplification of the formula. However, it is easily seen that the semi-algebraic set defined by the two definitions are equal up to homotopy equivalence.

The next proposition proved in [6] is important in the proof of Proposition 2.3; it relates up to p-equivalence the semi-algebraic set $\mathcal{J}_f^p(S)$ to the image of a closed, continuous semi-algebraic surjection $f : S \to L$.

Proposition 2.12. [6] *Let $f : S \to T$ a closed, continuous semi-algebraic surjection with S a closed and bounded semi-algebraic set. Then, for every $p \geq 0$, the map f induces a semi-algebraic p-equivalence $J(f) : \mathcal{J}_f^p(S) \to T$.*

We now define a thickened version of the semi-algebraic set $\mathcal{J}_f^p(S)$ defined above and prove that it is homotopy equivalent to $\mathcal{J}_f^p(S)$.

Definition 2.13 (The thickened semi-algebraic fibered join)**.** For $S \subset \mathbb{R}^k$ a semi-algebraic set contained in $B_k(0, R)$ defined by a $\mathcal{P}$-formula $\phi_{\mathcal{P}}$, $p \geq 1$, and $\varepsilon > 0$ define

$$
\begin{aligned}
\mathcal{J}_{f,\varepsilon}^p(S) = \{ & (\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}, \mathbf{a}) \in \mathbb{R}^{(p+1)(k+1)+\binom{p+1}{2}} \mid \\
& \Omega^R(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}) \wedge \Theta_1^{\varepsilon}(\mathbf{t}, \mathbf{a}) \wedge \Theta_2^{\Phi}(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}) \wedge \Theta_3(\mathbf{x}^0, \ldots, \mathbf{x}^p, \mathbf{t}, \mathbf{a}) \}
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega^R &:= \bigwedge_{i=0}^{p} (|\mathbf{X}^i|^2 \leq R^2) \wedge |\mathrm{T}|^2 \leq 4, \\
\Theta_1^{\varepsilon} &:= \sum_{i=0}^{p} T_i = 1 \wedge \sum_{1 \leq i < j \leq p} A_{ij}^2 \leq \varepsilon, \\
\Theta_2^{\Phi} &:= \bigwedge_{i=0}^{p} (T_i = 0 \vee \Phi(\mathbf{X}^i)), \\
\Theta_3^{f} &:= \bigwedge_{0 \leq i < j \leq p} (T_i = 0 \vee T_j = 0 \vee |f(\mathbf{X}^i) - f(\mathbf{X}^j)|^2 = A_{ij}).
\end{aligned}
\tag{2.2}
$$

Note that if S is closed and bounded over $\mathbb{R}$, so is $\mathcal{J}^p_{f,\varepsilon}(S)$.

The relation between $\mathcal{J}^p_f(S)$ and $\mathcal{J}^p_{f,\varepsilon}(S)$ is described in the following proposition.

Proposition 2.14. *For $p \in \mathbb{N}$, $f : S \to T$ semi-algebraic there exists $\varepsilon_0 > 0$ such that $\mathcal{J}^p_f(S)$ is homotopy equivalent to $\mathcal{J}^p_{f,\varepsilon}(S)$ for all $0 < \varepsilon \leq \varepsilon_0$.*

We prove the above proposition in Section 2.3 after proving a few preliminary results.

Proposition 2.15. *For $p \in \mathbb{N}$, $f : S \to T$ semi-algebraic, and $0 < \varepsilon \leq \varepsilon'$,*

$$\mathcal{J}^p_{f,\varepsilon}(S) \subseteq \mathcal{J}^p_{f,\varepsilon'}(S).$$

Moreover, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon' < \varepsilon_0$ the above inclusion induces a semi-algebraic homotopy equivalence.

The first part of above proposition is obvious from the definition of $\mathcal{J}^p_{f,\varepsilon}(S)$. The second part follows from the next lemma.

Lemma 2.16. *Let $T \subset \mathbb{R}^{k+1} \cap \{(\boldsymbol{x}, t) \in \mathbb{R}^{k+1} |\ t > 0\}$ be a semi-algebraic set. Suppose that $T_\varepsilon \subset T_{\varepsilon'}$ for all $0 < \varepsilon < \varepsilon'$, then there exists ε_0 such that for each $0 < \varepsilon < \varepsilon' \leq \varepsilon_0$ the inclusion map $T_\varepsilon \overset{i_{\varepsilon'}}{\hookrightarrow} T_{\varepsilon'}$ induces a semi-algebraic homotopy equivalence.*

Proof. We prove that there exists $\phi_{\varepsilon'} : T_{\varepsilon'} \to T_\varepsilon$ such that

$$\begin{aligned}
\phi_{\varepsilon'} \circ i_{\varepsilon'} : T_\varepsilon \to T_\varepsilon\ , &\quad \phi_{\varepsilon'} \circ i_{\varepsilon'} \simeq \mathrm{Id}_{T_\varepsilon},\\
i_{\varepsilon'} \circ \phi_{\varepsilon'} : T_{\varepsilon'} \to T_{\varepsilon'}, &\quad i_{\varepsilon'} \circ \phi_{\varepsilon'} \simeq \mathrm{Id}_{T_{\varepsilon'}}.
\end{aligned}$$

We first define $i_t : T_\varepsilon \hookrightarrow T_t$ and $\widehat{i}_t : T_t \hookrightarrow T_{\varepsilon'}$, and note that trivially $i_\varepsilon = \mathrm{Id}_{T_\varepsilon}$, $\widehat{i}_{\varepsilon'} = \mathrm{Id}_{T_{\varepsilon'}}$, and $i_{\varepsilon'} = \widehat{i}_\varepsilon$. Now, by Hardt triviality there exists $\varepsilon_0 > 0$, such that there is a definably trivial homeomorphism h which commutes with the projection π_{k+1}, *i.e.*, the following diagram commutes.

$$\begin{array}{ccc}
T_{\varepsilon_0} \times (0, \varepsilon_0] & \xrightarrow{\ h\ } & T \cap \{(\mathbf{x}, t)|\ 0 < t \leq \varepsilon_0\} \\
\downarrow{\scriptstyle \pi_{k+1}} & \swarrow{\scriptstyle \pi_{k+1}} & \\
(0, \varepsilon_0] & &
\end{array}$$

Define $F(\mathbf{x}, t, s) = h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, t), s)$. Note that $F(\mathbf{x}, t, t) = h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, t), t) = h(h^{-1}(\mathbf{x}, t)) = (\mathbf{x}, t)$. We define

$$\begin{aligned}
\phi_t &: T_t \to T_\varepsilon,\\
\phi_t(\mathbf{x}) &= \pi_{[1,k]} \circ F(\mathbf{x}, t, \varepsilon),\\
\widehat{\phi}_t &: T_{\varepsilon'} \to T_t,\\
\widehat{\phi}_t(\mathbf{x}) &= \pi_{[1,k]} \circ F(\mathbf{x}, \varepsilon', t)
\end{aligned}$$

and note that $\phi_{\varepsilon'} = \widehat{\phi}_\varepsilon$.

Finally, define

$$\begin{array}{ll} H_1(\cdot,t) & = \phi_t \circ i_t : T_\varepsilon \to T_\varepsilon, \\ H_1(\cdot,\varepsilon) & = \phi_\varepsilon \circ i_\varepsilon = \mathrm{Id}_{T_\varepsilon}, \\ H_1(\cdot,\varepsilon') & = \phi_{\varepsilon'} \circ i_{\varepsilon'}, \\ & \\ H_2(\cdot,t) & = \widehat{i}_t \circ \widehat{\phi}_t : T_{\varepsilon'} \to T_{\varepsilon'}, \\ H_2(\cdot,\varepsilon) & = \widehat{i}_\varepsilon \circ \widehat{\phi}_\varepsilon = i_{\varepsilon'} \circ \phi_{\varepsilon'}, \\ H_2(\cdot,\varepsilon') & = \widehat{i}_{\varepsilon'} \circ \widehat{\phi}_{\varepsilon'} = \mathrm{Id}_{T_{\varepsilon'}}. \end{array}$$

The semi-algebraic continuous maps H_1 and H_2 defined above give a semi-algebraic homotopy between the maps $\phi_{\varepsilon'} \circ i_{\varepsilon'} \simeq \mathrm{Id}_{T_\varepsilon}$ and $i_{\varepsilon'} \circ \phi_{\varepsilon'} \simeq \mathrm{Id}_{T_{\varepsilon'}}$, proving the required semi-algebraic homotopy equivalence.

□

Definition 2.17 (The thickened diagonal)**.** For a semi-algebraic set $S \subset \mathbb{R}^k$ contained in $B_k(0,R)$ defined by a $\mathcal{P}$-formula $\phi_\mathcal{P}$, $p \geq 1$, and $\varepsilon > 0$, define

$$\begin{aligned} \mathcal{D}^p_\varepsilon(S) = & \{(\mathbf{x}^0,\dots,\mathbf{x}^p,\mathbf{t},\mathbf{a}) \in \mathbb{R}^{(p+1)(k+1)+\binom{p+1}{2}} \mid \\ & \Omega^R(\mathbf{x}^0,\dots,\mathbf{x}^p,\mathbf{t}) \wedge \Theta_1(\mathbf{t},\mathbf{a}) \wedge \Theta_2^\Phi(\mathbf{x}^0,\dots,\mathbf{x}^p,\mathbf{t}) \wedge \Upsilon(\mathbf{x}^0,\dots,\mathbf{x}^p,\mathbf{t},\mathbf{a})\} \end{aligned}$$

where

$$\begin{aligned} \Omega^R &:= \bigwedge_{i=0}^p (|\mathbf{X}^i|^2 \leq R^2) \wedge |\mathrm{T}|^2 \leq 4, \\ \Theta_1^\varepsilon &:= \sum_{i=0}^p T_i = 1 \wedge \sum_{1\leq i<j\leq p} A_{ij}^2 \leq \varepsilon, \\ \Theta_2^\Phi &:= \bigwedge_{i=0}^p (T_i = 0 \vee \Phi(\mathbf{X}^i)), \\ \Upsilon &:= \bigwedge_{0\leq i<j\leq p} (T_i = 0 \vee T_j = 0 \vee |\mathbf{X}^i - \mathbf{X}^j|^2 = A_{ij}). \end{aligned} \tag{2.3}$$

Proposition 2.18. *Let $S \subset \mathbb{R}^k$ be a semi-algebraic set defined by a formula Φ having dense format bounded by (s,d,k), and division-free additive format bounded by (a,k). Then, $\mathcal{D}^p_\delta(S)$ is a semi-algebraic subset set of $\mathbb{R}^N$, defined by a formula with division-free additive format bounded by (M,N) and dense format bounded by $(M',d+1,N)$, where $M = (p+1)(k+a+3) + 2\binom{p+1}{2}(k+2)$, $M' = (p+1)(s+2) + 3\binom{p+1}{2} + 3$, and $N = (p+1)(k+1) + \binom{p+1}{2}$.*

Proof. We bound the division-free additive (resp. dense) format of the formulas $\Omega, \Theta_1^\varepsilon, \Theta_2^\Phi, \Upsilon$. Let

$$\begin{array}{ll} M_\Omega = (p+1)k + (p+1), & M'_\Omega = (p+1) + 1, \\ M_{\Theta_1^\varepsilon} = (p+1) + \binom{p+1}{2}, & M'_{\Theta_1^\varepsilon} = 2, \\ M_{\Theta_2^\Phi} = (p+1)(a+1), & M'_{\Theta_2^\Phi} = (p+1)(s+1), \\ M_\Upsilon = \binom{p+1}{2}(2k+2), & M'_\Upsilon = 3\binom{p+1}{2}. \end{array}$$

It is clear from Definition 2.17 that the division-free additive format (resp. dense format) of Ω is bounded by (M_Ω, N), $N = (p+1)(k+1) + \binom{p+1}{2}$ (resp. $(M'_\Omega, 2, N)$). Similarly, the division-free additive format (resp. dense format) of $\Theta_1^\varepsilon, \Theta_2^\Phi, \Upsilon$ is bounded by $(M_{\Theta_1^\varepsilon}, N), (M_{\Theta_2^\Phi}, N), (M_\Upsilon, N)$ (resp. $(M'_{\Theta_1}, 2, N)$, $(M'_{\Theta_2^\Phi}, d+1, N)$, $(M'_\Upsilon, 2, N)$). The division-free additive format (resp. dense format) of the formula defining $\mathcal{D}_\varepsilon^p$ above is thus bounded by

$$\left(M_\Omega + M_{\Theta_1^\varepsilon} + M_{\Theta_2^\Phi} + M_\Upsilon, N\right) \qquad (\text{resp.}(M'_\Omega + M'_{\Theta_1^\varepsilon} + M'_{\Theta_2^\Phi} + M'_\Upsilon, d+1, N)).$$

□

We now relate the thickened semi-algebraic fibered-join and the thickened diagonal using a sandwiching argument similar in spirit to that used in [19].

We will use a non-archimedean real closed extension of $\mathbb{R}$, namely the field of algebraic Puiseux series in the variable ε with coefficients in $\mathbb{R}$, which we will denote by $\mathbb{R}\langle\varepsilon\rangle$ (see [4, Chapter 2, Section 6]). The subring $\mathbb{R}\langle\varepsilon\rangle_b$ of elements of $\mathbb{R}\langle\varepsilon\rangle$ bounded by elements of $\mathbb{R}$ consists of Puiseux series with non-negative exponents.

Notation 2.19. We denote by $\lim_\varepsilon$ the ring homomorphism from $\mathbb{R}\langle\varepsilon\rangle_b$ to $\mathbb{R}$ which maps $\sum_{i\in\mathbb{N}} a_i\varepsilon^{i/q}$ to a_0.

Definition 2.20. We recall from [4] for a semi-algebraic subset $S \subseteq \mathbb{R}^k$ defined by a $\mathcal{P}$-formula Φ that $\mathrm{Ext}(S, \mathbb{R}\langle\varepsilon\rangle)$ the *extension of S to $\mathbb{R}\langle\varepsilon\rangle$* is the set defined by the same formula over $\mathbb{R}\langle\varepsilon\rangle$. That is,

$$\mathrm{Ext}(S, \mathbb{R}\langle\varepsilon\rangle) = \{\mathbf{x} \in \mathbb{R}\langle\varepsilon\rangle^k \mid \Phi(\mathbf{x})\}.$$

Given a semi-algebraic map $g : S \to T$ and a real closed extension $\mathbb{R}\langle\varepsilon\rangle$ we define $\mathrm{Ext}(g) : \mathrm{Ext}(S, R\langle\varepsilon\rangle) \to \mathrm{Ext}(T, R\langle\varepsilon\rangle)$ the *extension of g to $\mathbb{R}\langle\varepsilon\rangle$* to be the map having as a graph the set $\mathrm{Ext}(\Gamma(g), \mathbb{R}\langle\varepsilon\rangle)$, where $\Gamma(g) = \{(\mathbf{x}, \mathbf{y}) \mid g(\mathbf{y}) = \mathbf{x}\}$ is the *graph* of g.

The following lemma is easy to prove and we omit its proof (see [4]).

Lemma 2.21. *Let $S \subset \mathbb{R}^k$ be a closed and bounded semi-algebraic set. Then* $\lim_\varepsilon \mathrm{Ext}(S, \mathbb{R}\langle\varepsilon\rangle) = S$.

Lemma 2.22. *If $S \subset \mathbb{R}^{k+\ell}$ is a closed and bounded semi-algebraic set, then for any element $\mathbf{x} \in \mathbb{R}\langle\varepsilon\rangle^\ell$ which is bounded over $\mathbb{R}$, we have*

$$\lim_\varepsilon \mathrm{Ext}(S, \mathbb{R}\langle\varepsilon\rangle)_{\mathbf{x}} \subseteq S_{\lim_\varepsilon \mathbf{x}}.$$

Proof. Immediate from Lemma 2.21.

□

2.2.1. *Limits of one-parameter families.* In this section, we fix a bounded semi-algebraic set $T \subset \{(\mathbf{x}, \lambda) \in \mathbb{R}^{k+1} \mid \lambda > 0\}$ such that each fiber T_λ is closed and $T_\lambda \subseteq B_k(0, R)$ for some $R \in \mathbb{R}$ and all $\lambda > 0$.

In [19], the author proves

Proposition 2.23 ([19] Prop. 8)**.** *There exists $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ there exists a continuous semi-algebraic surjection $f_\lambda : T_\lambda \to \overline{T} \cap \pi_{k+1}^{-1}(0)$ such that the family of maps $\{f_\lambda \mid \lambda \in (0, \lambda_0)\}$ additionally satisfies the following two properties:*

(A)

$$\lim_{\lambda\to 0} \max_{\mathbf{x}\in T_\lambda} |\mathbf{x} - f_\lambda(\mathbf{x})| = 0,$$

and furthermore

(B) *for each* $\lambda, \lambda' \in (0, \lambda_0)$, $f_\lambda = f_{\lambda'} \circ h$ *for some semi-algebraic homeomorphism* $h : X_\lambda \to X_{\lambda'}$.

We give a different proof of the above proposition than the one in [19] in what follows. Specifically, we define a different semi-algebraic map than the one which appears in [19], which avoids explicit use of triangulation of the set T.

We now define the function f_λ for $\lambda > 0$ sufficiently small. By Hardt's triviality theorem applied to π_{k+1} and the semi-algebraic sets $(\mathbb{R}^{k+1}, B_{k+1}(0, R), T)$, there is a semi-algebraic set $A \subseteq \overline{B_k(0, R)}$, $\lambda_0 > 0$, and $h : A \times (0, \lambda_0] \to T \cap \pi_{k+1}^{-1}((0, \lambda_0])$ a semi-algebraic homeomorphism which is definably trivial over the map π_{k+1} (that is $\pi_{k+1} \circ h = \pi_{k+1}$).

We define a semi-algebraic map $f_\lambda(\mathbf{x})$ for $\mathbf{x} \in T_\lambda$ by

$$\begin{aligned} f_\lambda &: T_\lambda \to \mathbb{R}^{k+1} \\ f_\lambda(\mathbf{x}) &= \lim_{t \to 0} h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t). \end{aligned} \tag{2.4}$$

We define for $(\mathbf{x}, \lambda) \in T$ the semi-algebraic map

$$\begin{aligned} F_{(\mathbf{x},\lambda)} &: [0, \lambda_0] \to \mathbb{R}^{k+1} \\ t &\mapsto \begin{cases} h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t) & \text{if } t \neq 0 \\ f_\lambda(\mathbf{x}) & \text{if } t = 0. \end{cases} \end{aligned} \tag{2.5}$$

For any fixed $(\mathbf{x}, \lambda)$, we will call the graph $\Gamma(F_{(\mathbf{x},\lambda)}) = \{(\mathbf{y}, t) | \; F_{(\mathbf{x},\lambda)}(t) = \mathbf{y}\}$ of the map $F_{(\mathbf{x},\lambda)}$ the *trajectory* of F passing through the point $(\mathbf{x}, \lambda)$. Note that each such trajectory is a semi-algebraic curve contained in T for $t > 0$, and contained in $\overline{T} \cap \pi_{k+1}^{-1}(0)$ for $t = 0$.

Note that if $(\mathbf{x}', \lambda')$ belongs to $\Gamma(F_{(\mathbf{x},\lambda)})$ then $F_{(\mathbf{x},\lambda)}(t) = F_{(\mathbf{x}',\lambda')}(t)$ for all $t \in (0, \lambda_0]$ since h is a homeomorphism, and hence the functions are equal at $t = 0$ as well. Also, a point $(\mathbf{x}, \lambda) \in T$ belongs to exactly one trajectory. Thus two distinct trajectories do not have a point in common for $(\mathbf{x}, \lambda)$ with $\lambda \in (0, \lambda_0)$, but may intersect at $t = 0$.

We now define the semi-algebraic map F by

$$\begin{aligned} F &: T \times [0, \lambda_0] \to \mathbb{R}^{k+1} \\ (\mathbf{x}, \lambda, t) &\mapsto F_{(\mathbf{x},\lambda)}(t) \end{aligned} \tag{2.6}$$

and prove

Lemma 2.24. *For* $(\boldsymbol{x}, \lambda) \in \mathbb{R}\langle\varepsilon\rangle_b^{k+1}$, $\lambda > 0$ *and not infinitesimal, we have*

$$\lim_{\varepsilon} \operatorname{Ext}(\Gamma(F), \mathbb{R}\langle\varepsilon\rangle)_{(\mathbf{x},\lambda)} \subseteq \Gamma(F)_{\lim_\varepsilon(\mathbf{x},\lambda)}.$$

Proof. Apply Lemma 2.22 to $\Gamma(F)$ which is a closed and bounded subset of $\mathbb{R}^{2k+3}$.

□

Proof of Proposition 2.23. Let $T \subset \mathbb{R}^{k+1}$ and $\{T_\lambda\}_{\lambda>0}$ be as in the statement of the proposition. By Hardt's triviality theorem applied to the projection map $\pi_{k+1} : \mathbb{R}^{k+1} \to \mathbb{R}$ and the semi-algebraic sets $(\mathbb{R}^{k+1}, B_{k+1}(0, R), T)$ we have that there exists a semi-algebraic set $A \subset \mathbb{R}^k$ and a $\lambda_0 > 0$ such that there is a homeomorphism

$h : A \times (0, \lambda_0] \to T \cap \pi_{k+1}^{-1}(0, \lambda_0)$ which is definably trivial over the map π_{k+1}; that is, $\pi_{k+1} \circ h = \pi_{k+1}$. Define $f_\lambda(\mathbf{x})$ for $\mathbf{x} \in T_\lambda$ as in Equation (2.4) as

$$f_\lambda(\mathbf{x}) = \lim_{t \to 0} h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t).$$

We need to show several things: (0) the map $f_\lambda : T_\lambda \to \overline{T} \cap \pi_{k+1}^{-1}(0)$ is well-defined and the co-domain of the map is $\overline{T} \cap \pi_{k+1}^{-1}(0)$, the maps f_λ are (1) continuous and (2) surjective, and the family of maps are (3) close to the identity and (4) they can be obtained from each other by composition with a homeomorphism.

(0) Let $\lambda \in (0, \lambda_0)$ and $\mathbf{x} \in T_\lambda$. We show that $\lim_{t \to 0} h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t) \in \overline{T} \cap \pi_{k+1}^{-1}(0)$. Using that the map π_{k+1} is continuous and that $\pi_{k+1} \circ h = \pi_{k+1}$ we have the following.

$$\begin{aligned} \pi_{k+1}(\lim_{t \to 0} h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t)) &= \lim_{t \to 0} \pi_{k+1} \circ h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t) \\ &= \lim_{t \to 0} \pi_{k+1}(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t) \\ &= \lim_{t \to 0} t = 0. \end{aligned}$$

So, $\pi_{k+1} \circ f_\lambda(\mathbf{x}) = 0$. To see that $f_\lambda(\mathbf{x}) \in \overline{T}$, we just use the definition of closure, $\overline{T} = \{\mathbf{z} \in \mathbb{R}^N | \ (\forall \varepsilon)(\varepsilon > 0 \implies (\exists \mathbf{y})(\mathbf{y} \in T \wedge |\mathbf{z} - \mathbf{y}| < \varepsilon))\}$. Specifically, if $\delta > 0$ choose t_0 depending on δ, $\mathbf{x}$, and λ which satisfies

$$|\lim_{t \to 0} h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t) - h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t_0)| < \delta,$$

which it is clear we can do. Since $h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t_0) \in T$ and $\lim_{t \to 0} h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t) = f_\lambda(\mathbf{x})$, we have $f_\lambda(\mathbf{x}) \in \overline{T}$. So, $f_\lambda(\mathbf{x}) \in \overline{T} \cap \pi_{k+1}^{-1}(0)$.

(1) Let $\mathbf{x} \in T_\lambda$, $\varepsilon > 0$ and it suffices to show that for some $\delta > 0$ if $\mathbf{y} \in T_\lambda$ satisfies $|\mathbf{x} - \mathbf{y}| < \delta$ then $|f_\lambda(\mathbf{x}) - f_\lambda(\mathbf{y})| < \varepsilon$. Since h is continuous, we can pick δ_1 such that $|(a, \lambda) - (b, \lambda)| < \delta_1$ implies $|h(a, \lambda) - h(b, \lambda)| < \frac{\varepsilon}{2}$. Since $h^{-1} : T \cap \pi_{k+1}^{-1}(\lambda) \to A \times \lambda$ is continuous, we can pick δ such that $|(\mathbf{x}, \lambda) - (\mathbf{y}, \lambda)| < \delta$ implies $|h^{-1}(\mathbf{x}, \lambda) - h^{-1}(\mathbf{y}, \lambda)| < \delta_1$. In particular for $\mathbf{y} \in T_\lambda$ satisfying $|\mathbf{x} - \mathbf{y}| < \delta$ we have, for each $t > 0$, that $|(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t) - (\pi_{[1,k]} \circ h^{-1}(\mathbf{y}, \lambda), t)| < \delta_1$ as well, and hence

$$|\ h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t) - h(\pi_{[1,k]} \circ h^{-1}(\mathbf{y}, \lambda), t)\ | < \tfrac{\varepsilon}{2}.$$

Thus, taking the limit as $t > 0$ tends to zero we have

$$\lim_{t \to 0}\ |\ h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t) - h(\pi_{[1,k]} \circ h^{-1}(\mathbf{y}, \lambda), t)\ | \leq \tfrac{\varepsilon}{2}.$$

Using the fact that the norm function is continuous, we may take the limit inside and obtain

$$|\ \lim_{t \to 0} h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t) - \lim_{t \to 0} h(\pi_{[1,k]} \circ h^{-1}(\mathbf{y}, \lambda), t)\ | = |f_\lambda(\mathbf{x}) - f_\lambda(\mathbf{y})| \leq \tfrac{\varepsilon}{2}.$$

(2) In this part we consider the real closed extension $\mathbb{R}\langle\varepsilon\rangle$ of $\mathbb{R}$. Let $\lambda > 0$ and $\mathbf{y} \in T_0$; note that $(\mathbf{y}, 0) \in \overline{T} \cap \pi_{k+1}^{-1}(0)$. We need to find $\mathbf{x} \in T_\lambda$ such that

$f_\lambda(\mathbf{x}) = (\mathbf{y}, 0)$. Let $(\mathbf{y}, 0) \in \overline{\mathrm{Ext}(T, \mathbb{R}\langle\varepsilon\rangle)}$. Using the definition of closure, there exists $(\mathbf{z}_1, \lambda_1) \in \mathrm{Ext}(T, \mathbb{R}\langle\varepsilon\rangle)$ such that

$$|(\mathbf{y}, 0) - (\mathbf{z}_1, \lambda_1)| < \varepsilon.$$

Pick $\lambda_2 > 0$ that is not infinitesimal, and let $(\mathbf{z}_2, \lambda_2) \in \mathrm{Ext}(\Gamma(F), \mathbb{R}\langle\varepsilon\rangle)_{(\mathbf{z}_1,\lambda_1)}$.

By Lemma 2.24

$$\lim_\varepsilon \mathrm{Ext}(\Gamma(F), \mathbb{R}\langle\varepsilon\rangle)_{(\mathbf{z}_2,\lambda_2)} \subseteq \Gamma(F)_{\lim_\varepsilon(\mathbf{z}_2,\lambda_2)}.$$

Hence,

$$\lim_\varepsilon (\mathbf{z}_1, \lambda_1) = (\mathbf{y}, 0) \in \Gamma(F)_{\lim_\varepsilon(\mathbf{z}_2,\lambda_2)}.$$

Let $(\mathbf{x}, \lambda) \in \Gamma(F)_{\lim_\varepsilon(\mathbf{z}_2,\lambda_2)}$ for our choice of $\lambda > 0$. It is now easy to see that $\mathbf{x} \in T_\lambda$ satisfies $f_\lambda(\mathbf{x}) = \mathbf{y}$ as desired.

(3) We prove that the family of continuous, surjective maps $\{f_\lambda\}_{\lambda>0}$ is close to the identity in that

$$\lim_{\lambda\to 0} \max_{\mathbf{x}\in T_\lambda} |(\mathbf{x}, \lambda) - f_\lambda(\mathbf{x})| = 0.$$

Note that $f_\lambda(\mathbf{x}) = F(\mathbf{x}, \lambda, 0)$ and that the max function is semi-algebraic. Set

$$p(\mathbf{x}, \lambda) = |\mathbf{x} - F(\mathbf{x}, \lambda, 0)|,$$

and

$$g(\lambda) = \max_{\mathbf{x}\in T_\lambda} p(\mathbf{x}, \lambda).$$

So g, p are semi-algebraic. In particular, letting $T = \{(\mathbf{x}, \lambda) \in \mathbb{R}^{k+1} | \; \Phi(\mathbf{x}, \lambda)\}$ and noting that $g(\lambda)$ is continuous we see that $g(\lambda) = c$ if and only if the following formula is true.

$$(\forall \mathbf{x})(\Phi(\mathbf{x}, \lambda) \implies (p(\mathbf{x}, \lambda) \le c \wedge (\exists \mathbf{y})(\Phi(\mathbf{y}, \lambda) \wedge p(\mathbf{y}, \lambda) = c))).$$

The same is true when we consider the extensions $\mathrm{Ext}(T, \mathbb{R}\langle\varepsilon\rangle)$, $\mathrm{Ext}(g)$, and $\mathrm{Ext}(p)$.

It now suffices to show that $\mathrm{Ext}(g)(\varepsilon)$ is infinitesimal. Let $\mathbf{y}_\varepsilon \in \mathrm{Ext}(T, \mathbb{R}\langle\varepsilon\rangle)_\varepsilon$ such that $\mathrm{Ext}(p)(\mathbf{y}_\varepsilon, \varepsilon) = \mathrm{Ext}(g)(\varepsilon)$. Seeking a contradiction, suppose that $\mathrm{Ext}(p)(y_\varepsilon, \varepsilon)$ is not infinitesimal. This means that

$$\lim_\varepsilon (y_\varepsilon, \varepsilon) \neq \lim_\varepsilon \mathrm{Ext}(F)(y_\varepsilon, \varepsilon, 0).$$

Let $(y_\lambda, \lambda) \in \mathrm{Ext}(\Gamma(F), \mathbb{R}\langle\varepsilon\rangle)_{(y_\varepsilon,\varepsilon)}$ with $\lambda > 0$ not infinitesimal. Set $(y_1, 0) = \lim_\varepsilon (y_\varepsilon, \varepsilon)$ and $(y_2, 0) = \lim_\varepsilon \mathrm{Ext}(F)(y_\varepsilon, \varepsilon, 0)$, and note $y_1 \neq y_2$ and both $(y_1, 0)$, $(y_2, 0)$ belong to $\lim_\varepsilon \mathrm{Ext}(\Gamma(F), \mathbb{R}\langle\varepsilon\rangle)_{(y_\varepsilon,\varepsilon)}$. Hence, by Lemma 2.24 and our choice of (y_λ, λ), we have that $(y_1, 0), (y_2, 0)$ both belong to $\Gamma(F)_{\lim_\varepsilon(y_\lambda,\lambda)}$, which is a contradiction since every such trajectory intersects the λ-axis at exactly one point (since $F(\mathbf{x}, \lambda, 0)$ is defined as a limit).

(4) Finally, we show that f_λ can be obtained from $f_{\lambda'}$ by composition with a semi-algebraic homeomorphism. This follows immediately from the fact that $h : A \times (0, \lambda_0] \to \pi_{k+1}^{-1}((0, \lambda_0]) \cap T$ is a fiber preserving homeomorphism. In particular, if we restrict h to $A \times \{\lambda\}$ we get a semi-algebraic homeomorphism

$$h_1 : A \times \{\lambda\} \to \pi_{k+1}^{-1}(\lambda) \cap T.$$

Similarly, we obtain h_2 restricted to $A \times \{\lambda'\}$, while the near-identity map $id : A \times \{\lambda\} \to A \times \{\lambda'\}$, $(\mathbf{x}, \lambda) \overset{id}{\mapsto} (\mathbf{x}, \lambda')$ is a homeomorphism.

Finally, the restriction of the projection map $\pi^{\lambda}_{[1,k]} : \pi^{-1}_{k+1}(\lambda) \cap T \to T_\lambda$ is a semi-algebraic homeomorphism for both λ and λ'. Hence, $\pi^{\lambda'}_{[1,k]} \circ h_2 \circ \mathrm{Id} \circ h_1^{-1} \circ (\pi^{\lambda}_{k+1})^{-1} : T_\lambda \to T_{\lambda'}$ is a semi-algebraic homeomorphism and, following the definitions, $f_{\lambda'} \circ (\pi^{\lambda'}_{[1,k]} \circ h_2 \circ \mathrm{Id} \circ h_1^{-1} \circ (\pi^{\lambda}_{k+1})^{-1}) = f_\lambda$ as desired.

□

Proposition 2.25. [19] *There exists λ_1 satisfying $0 < \lambda_1 \leq \lambda_0$ and semi-algebraic functions $\delta_0(\lambda)$ and $\delta_1(\lambda)$ defined for each $\lambda \in (0, \lambda_1)$ such that $\lim_{\lambda\to 0} \delta_0(\lambda) = 0$, $\lim_{\lambda\to 0} \delta_1(\lambda) \neq 0$, and such that for all $0 < \delta_0(\lambda) < \delta' < \delta < \delta_1(\lambda)$ the inclusion $\mathcal{D}^p_{\delta'}(T_\lambda) \hookrightarrow \mathcal{D}^p_{\delta}(T_\lambda)$ induces a semi-algebraic homotopy equivalence.*

The above proposition is adapted from Proposition 20 in [19] and the proof is identical after replacing $D^p_\lambda(\delta)$ (defined in [19]) with the semi-algebraic set $\mathcal{D}^p_\delta(T_\lambda)$ defined above (Definition 2.17).

As in [19], define for $p \in \mathbb{N}$,

$$\eta_p(\lambda) = p(p+1)\left(4R \max_{\mathbf{x}\in T_\lambda} |\mathbf{x} - f_\lambda(\mathbf{x})| + 2\left(\max_{\mathbf{x}\in T_\lambda} |\mathbf{x} - f_\lambda(\mathbf{x})|\right)^2\right). \tag{2.7}$$

Note that for $q \leq p$ we have $\eta_q(\lambda) \leq \eta_p(\lambda)$, and that $\lim_{\lambda\to 0} \eta_p(\lambda) = 0$ for each $p \in \mathbb{N}$ since f_λ approaches the identity (Prop. 2.23 A).

Define for $\overline{\mathbf{x}} = (\mathbf{x}^0, \ldots, \mathbf{x}^p) \in \mathbb{R}^{(p+1)k}$ the sum $\rho_p(\overline{\mathbf{x}})$ as

$$\rho_p(\mathbf{x}^0, \ldots, \mathbf{x}^p) = \sum_{1\leq i<j\leq p} |\mathbf{x}^i - \mathbf{x}^j|^2.$$

A special case of this sum corresponding to all $t_i \neq 0$ appears in the formula Υ^ε_1 of Definition 2.17 after making the replacement $a_{ij} = |\mathbf{x}^i - \mathbf{x}^j|$. The next lemma is taken from [19] to which we refer the reader for the proof.

Lemma 2.26 ([19] Lemma 21)**.** *Given $\eta_p(\lambda)$ and $f_\lambda : T_\lambda \to T$ as above, we have*

$$|\sum_{i<j} |f_\lambda(\mathbf{x}^i) - f_\lambda(\mathbf{x}^j)|^2 - \sum_{i<j} |\mathbf{x}^i - \mathbf{x}^j|^2| \;\leq\; \eta_p(\lambda),$$

and in particular

$$\rho_p(\mathbf{x}^0, \ldots, \mathbf{x}^p) \leq \rho_p(f_\lambda(\mathbf{x}^0), \ldots, f_\lambda(\mathbf{x}^p)) + \eta_p(\lambda) \leq \rho_p(\mathbf{x}^0, \ldots, \mathbf{x}^p) + 2\eta_p(\lambda).$$

The next proposition follows immediately from the previous lemma, Definition 2.13, and Definition 2.17.

Proposition 2.27. [19] *If $\lambda \in (0, \lambda_0)$ is such that $\varepsilon, \varepsilon + \eta_p(\lambda), \varepsilon + 2\eta_p(\lambda) \in (0, \varepsilon_0)$ then*

$$\mathcal{J}^p_{f_\lambda,\varepsilon}(T_\lambda) \subseteq \mathcal{D}^p_{\varepsilon+\eta_p(\lambda)}(T_\lambda) \subseteq \mathcal{J}^p_{f_\lambda,\varepsilon+2\eta_p(\lambda)}(T_\lambda).$$

The statement and proof of the following proposition are adapted from [19].

Proposition 2.28. [19] *For any $p \in \mathbb{N}$, there exists $\lambda \in (0, \lambda_0)$, $\varepsilon \in (0, \varepsilon_0)$ and $\delta > 0$ such that $\mathcal{D}^p_\delta(T_\lambda) \overset{i}{\hookrightarrow} \mathcal{J}^p_{f_\lambda,\varepsilon}(T_\lambda)$ and such that this inclusion induces a homotopy equivalence*

$$\mathcal{D}^p_\delta(T_\lambda) \simeq \mathcal{J}^p_{f_\lambda,\varepsilon}(T_\lambda).$$

Proof. We first describe how to choose $\lambda \in (0, \lambda_0)$, $\varepsilon, \varepsilon' \in (0, \varepsilon_0)$ and $\delta, \delta' \in (\delta_0(\lambda), \delta_1(\lambda))$ so that

$$\mathcal{D}^p_\delta(T_\lambda) \subseteq \mathcal{J}^p_{f_\lambda,\varepsilon}(T_\lambda) \subseteq \mathcal{D}^p_{\delta'}(T_\lambda) \subseteq \mathcal{J}^p_{f_\lambda,\varepsilon'}(T_\lambda),$$

and secondly we show how this induces isomorphisms between the homotopy groups.

Since the limit of $\delta_1(\lambda) - \delta_0(\lambda)$ is not zero for $0 < \lambda < \lambda_0$ and λ tending to zero, while the limits of $\eta_p(\lambda)$ and $\delta_0(\lambda)$ are zero (by Prop. 2.23 A, Prop. 2.25 resp.), we can choose $0 < \lambda < \lambda_0$ which simultaneously satisfies

$$2\eta_p(\lambda) < \tfrac{\delta_1(\lambda)-\delta_0(\lambda)}{2} \qquad \text{and} \qquad \delta_0(\lambda) + 4\eta_p(\lambda) < \varepsilon_0.$$

Then, since $\delta_0(\lambda) > 0$ we have the following two inequalities

$$2\eta_p(\lambda) < \tfrac{\delta_1(\lambda)+\delta_0(\lambda)}{2} \qquad \text{and} \qquad 2\eta_p(\lambda) < \delta_1(\lambda) - \delta_0(\lambda).$$

Set $\delta = \delta_0 + \eta_p(\lambda)$, $\varepsilon = \delta' + 2\eta_p(\lambda)$, $\delta' = \delta + 3\eta_p(\lambda)$, and $\varepsilon' = \delta + 4\eta_p(\lambda)$. ¿From Proposition 2.27 we have the following inclusions,

$$\mathcal{D}^p_\delta(T_\lambda) \overset{i}{\hookrightarrow} \mathcal{J}^p_{f_\lambda,\varepsilon}(T_\lambda) \overset{j}{\hookrightarrow} \mathcal{D}^p_{\delta'}(T_\lambda) \overset{k}{\hookrightarrow} \mathcal{J}^p_{f_\lambda,\varepsilon'}(T_\lambda).$$

Furthermore, we have that $\delta, \delta' \in (\delta_0(\lambda), \delta_1(\lambda))$ and that $\varepsilon, \varepsilon' \in (0, \varepsilon_0)$, and so we have that both $j \circ i$ and $k \circ j$ induce semi-algebraic homotopy equivalences (Prop. 2.25, Prop. 2.15 resp.). This gives rise to a diagram between the homotopy groups.

$$\begin{array}{ccccc}
\pi_*(\mathcal{D}^p_\delta(T_\lambda)) & \xrightarrow[\cong]{(j\circ i)_*} & \pi_*(\mathcal{D}^p_{\delta'}(T_\lambda)) & & \\
 & {\scriptstyle i_*}\searrow \quad \nearrow{\scriptstyle j_*} & & \searrow{\scriptstyle k_*} & \\
 & \pi_*(\mathcal{J}^p_{f_\lambda,\varepsilon}(T_\lambda)) & \xrightarrow[\cong]{(k\circ j)_*} & \pi_*(\mathcal{J}^p_{f_\lambda,\varepsilon'}(T_\lambda)) &
\end{array}$$

Since $(j \circ i)_* = j_* \circ i_*$, the surjectivity of $(j \circ i)_*$ implies that j_* is surjective, and similarly $(k \circ j)_*$ is injective means that j_* is injective. Hence, j_* is an isomorphism as required.

This implies that the inclusion map between $\mathcal{D}^p_\delta(T_\lambda)$ and $\mathcal{J}^p_{f_\lambda,\varepsilon}(T_\lambda)$ is a weak homotopy equivalence (see [18, pp. 181]). Since both spaces have the structure of a finite CW-complex, every weak equivalence is in fact a homotopy equivalence. □

2.3. **Proof of Proposition 2.3.** We recall the Equations 2.5 and 2.6 using the fact that $f_\lambda : T_\lambda \to \overline{T} \cap \pi^{-1}_{k+1}(0)$.

$$\begin{aligned}
F_{(\mathbf{x},\lambda)} : [0, \lambda_0] &\to \overline{T} \\
t &\mapsto \begin{cases} h(\pi_{[1,k]} \circ h^{-1}(\mathbf{x}, \lambda), t) & \text{if } t \neq 0 \\ f_\lambda(\mathbf{x}) & \text{if } t = 0. \end{cases} \\
F : T \times [0, \lambda_0] &\to \overline{T} \\
(\mathbf{x}, \lambda, t) &\mapsto F_{(\mathbf{x},\lambda)}(t)
\end{aligned}$$

Lemma 2.29. *Let $T \subset \mathbb{R}^{k+1} \cap \{t > 0\}$ such that each T_t is closed and $T_t \subseteq B_k(0, R)$ for $t > 0$. Suppose further that for all $0 < t \leq t'$ we have $T_t \subseteq T_{t'}$. Then,*

$$\bigcap_{t>0} T_t = \pi_{[1,k]}\left(\overline{T} \cap \pi^{-1}_{k+1}(0)\right).$$

Furthermore, denoting $T_0 := \pi_{[1,k]}\left(\overline{T} \cap \pi_{k+1}^{-1}(0)\right)$ *there exists* $\varepsilon_0 > 0$ *such that for all* ε *satisfying* $0 < \varepsilon \leq \varepsilon_0$ *we have that* T_0 *is semi-algebraically homotopy equivalent to* T_ε*.*

Proof. We first prove $\bigcap_{t>0} T_t = \pi_{[1,k]}\left(\overline{T} \cap \pi_{k+1}^{-1}(0)\right)$ by proving both set inclusions.

Let $\mathbf{x} \in \cap_{t>0} T_t$, so $(\mathbf{x}, t) \in T$ for all $t \in \mathbb{R}$, $t > 0$. We have $\lim_{t \to 0^+}(\mathbf{x}, t) = (\mathbf{x}, 0) \in \overline{T}$. So, $\mathbf{x} \in T_0$.

For the other inclusion, let $\mathbf{x} \in T_0$ and $t_0 \in \mathbb{R}$ with $t_0 > 0$. It suffices to show that $\mathbf{x} \in T_{t_0}$. Consider the real closed extension $\mathbb{R}\langle\varepsilon\rangle$. Since $\mathbf{x} \in T_0$, there exists $(\mathbf{x}_\varepsilon, t_\varepsilon) \in \mathbb{R}\langle\varepsilon\rangle^{k+1}$ such that

$$(\mathbf{x}_\varepsilon, t_\varepsilon) \in \operatorname{Ext}(T, \mathbb{R}\langle\varepsilon\rangle) \cap B_{k+1}((\mathbf{x}, 0), \varepsilon).$$

Note that $\lim_\varepsilon \mathbf{x}_\varepsilon = \mathbf{x}$ and $\lim_\varepsilon t_\varepsilon = 0$. It follows from the assumption that $T_t \subseteq T_{t'}$ for all $t, t' \in \mathbb{R}$, $0 < t \leq t'$, and properties of $\operatorname{Ext}(\cdot, \mathbb{R}\langle\varepsilon\rangle)$ (see Definition 2.20) that

$$\operatorname{Ext}(T, \mathbb{R}\langle\varepsilon\rangle)_{t_\varepsilon} \subseteq \operatorname{Ext}(T, \mathbb{R}\langle\varepsilon\rangle)_{t_0},$$

since t_ε being infinitesimal is smaller than $t_0 \in \mathbb{R}$. Hence, since $\mathbf{x}_\varepsilon \in \operatorname{Ext}(T, \mathbb{R}\langle\varepsilon\rangle)_{t_\varepsilon}$ we have $\mathbf{x}_\varepsilon \in \operatorname{Ext}(T, \mathbb{R}\langle\varepsilon\rangle)_{t_0}$. Finally,

$$\operatorname{Ext}(T, \mathbb{R}\langle\varepsilon\rangle)_{t_0} = \operatorname{Ext}(T_{t_0}, \mathbb{R}\langle\varepsilon\rangle),$$

and T_{t_0} is closed and bounded. Using Lemma 2.21 we have $\lim_\varepsilon \operatorname{Ext}(T_{t_0}, \mathbb{R}\langle\varepsilon\rangle) = T_{t_0}$, and so $\mathbf{x} \in T_{t_0}$.

We now prove the second part of the lemma. We follow the proof of Lemma 2.16 except replace ε by 0 and ε' by ε. Clearly, $T_0 \subseteq T_t$ for all $t > 0$ (since it is an intersection). Define $i_t : T_0 \hookrightarrow T_t$ and $\widehat{i}_t : T_t \to T_\varepsilon$ for $t \geq 0$ and $t \leq \varepsilon$ respectively and note that $i_0 = \mathrm{Id}_{T_0}$ and $\widehat{i}_\varepsilon = \mathrm{Id}_{T_\varepsilon}$. Define

$$\begin{aligned}
&\phi_t : T_t \to T_0\\
&\phi_t(\mathbf{x}) = \begin{cases} \pi_{[1,k]} \circ F(\mathbf{x}, t, 0) & \text{if } t \neq 0 \\ \mathbf{x} & \text{if } t = 0 \end{cases}\\
&\widehat{\phi}_t : T_\varepsilon \to T_t\\
&\widehat{\phi}_t(\mathbf{x}) = \pi_{[1,k]} \circ F(\mathbf{x}, \varepsilon, t)
\end{aligned}$$

and note that trivially $\phi_\varepsilon = \widehat{\phi}_0$. We are now ready to give the desired semi-algebraic homotopy maps. Define

$$\begin{aligned}
H_1(\cdot, t) &= \phi_t \circ i_t : T_0 \to T_0,\\
H_1(\cdot, 0) &= \phi_0 \circ i_0 = \mathrm{Id}_{T_0},\\
H_1(\cdot, \varepsilon) &= \phi_\varepsilon \circ i_\varepsilon,\\
\\
H_2(\cdot, t) &= \widehat{i}_t \circ \widehat{\phi}_t : T_\varepsilon \to T_\varepsilon,\\
H_2(\cdot, 0) &= \widehat{i}_0 \circ \widehat{\phi}_0 = i_\varepsilon \circ \phi_\varepsilon,\\
H_2(\cdot, \varepsilon) &= \widehat{i}_\varepsilon \circ \widehat{\phi}_\varepsilon = \mathrm{Id}_{T_\varepsilon}.
\end{aligned}$$

Note that the homotopy $H_1 : T_0 \times [0, \varepsilon] \to T_0$ is continuous since $f_t(\mathbf{x}) = F(\mathbf{x}, t, 0)$ is close to the identity in the sense of Proposition 2.23. This proves that T_0 is semi-algebraically homotopy equivalent to T_ε.

□

Lemma 2.30. *For $p \in \mathbb{N}$, $f : S \to T$ semi-algebraic we have*

$$\mathcal{J}_f^p(S) = \bigcap_{\varepsilon>0} \mathcal{J}_{f,\varepsilon}^p(S).$$

Proof. Obvious from Definitions 2.10 and 2.13.

□

Proof of Proposition 2.14. The set $T = \{(\mathbf{x}, t) \in \mathbb{R}^{k+1} | \, t > 0 \wedge \mathbf{x} \in \mathcal{J}_{f,t}^p(S)\}$ satisfies the conditions of Lemma 2.29. The proposition now follows from Lemma 2.29 and Lemma 2.30.

□

Proof of Proposition 2.3. Let $T \subset \mathbb{R}^{k+1} \cap \{t > 0\}$ such that T_t is closed and $T_t \subseteq B_k(0, R)$ for some $R \in \mathbb{R}$ and all $t > 0$. Applying Proposition 2.28 we have that for some $\lambda \in (0, \lambda_0)$ and $\varepsilon \in (0, \varepsilon_0)$ (see Proposition 2.15 and Proposition 2.23 for definitions of ε_0 and λ_0 respectively) the sets $\mathcal{D}_\delta^p(T_\lambda) \simeq \mathcal{J}_{f_\lambda,\varepsilon}^p(T_\lambda)$ are homotopy equivalent. Also, by Proposition 2.14 the sets $\mathcal{J}_{f_\lambda,\varepsilon}^p(T_\lambda) \simeq \mathcal{J}_{f_\lambda}^p(T_\lambda)$ are semi-algebraically homotopy equivalent. By Proposition 2.12 and Proposition 2.23 the map $J(f_\lambda) : \mathcal{J}_{f_\lambda}^p(T_\lambda) \twoheadrightarrow T_0$ induces a p-equivalence.

Thus we have the following sequence of homotopy equivalences and p-equivalence.

$$\mathcal{D}_\delta^p(T_\lambda) \simeq \mathcal{J}_{f_\lambda,\varepsilon}^p(T_\lambda) \simeq \mathcal{J}_{f_\lambda}^p(T_\lambda) \xrightarrow{\sim}_p T_0 \tag{2.8}$$

The first homotopy equivalence follows from Proposition 2.28, the second from Proposition 2.14, and the last p-equivalence is a consequence of Propositions 2.12 and 2.23. The bound on the format of the formula defining $\mathcal{D}^p := \mathcal{D}_\delta^p(T_\lambda)$ follows from Proposition 2.18. This finishes the proof.

□

Proof of Theorem 2.1. The theorem follows directly from Proposition 2.3, Theorem 1.5 and Proposition 2.4 after choosing $p = k + 1$.

□

3. Proofs of Theorem 1.9 and Theorem 1.12

3.1. **Algebraic preliminaries.** We start by proving a lemma that provides a slightly different characterization of division-free additive complexity from that given in Definition 1.6. Roughly speaking the lemma states that any given additive representation of a given polynomial P can be modified without changing its length to another additive representation of P in which any negative exponents occur only in the very last step. This simplification will be very useful in what follows.

More precisely, we prove

Lemma 3.1. *For any $P \in \mathbb{R}[\mathbf{X}]$ and $a \in \mathbb{N}$ we have P has additive complexity at most a if and only if there exists a sequence of equations (*)*

(i) $Q_1 = u_1 X_1^{\alpha_{11}} \cdots X_k^{\alpha_{1k}} + v_1 X_1^{\beta_{11}} \cdots X_k^{\beta_{1m}}$,
where $u_1, v_1 \in \mathbb{R}$, and $\alpha_{11}, \ldots, \alpha_{1k}, \beta_{11}, \ldots, \beta_{1k} \in \mathbb{N}$;

(ii) $Q_j = u_j X_1^{\alpha_{j1}} \cdots X_k^{\alpha_{jk}} \prod_{1 \le i \le j-1} Q_i^{\gamma_{ji}} + v_j X_1^{\beta_{j1}} \cdots X_k^{\beta_{jk}} \prod_{1 \le i \le j-1} Q_i^{\delta_{ji}}$,
where $1 < j \le a$, $u_j, v_j \in \mathbb{R}$, *and* $\alpha_{j1}, \ldots, \alpha_{jk}, \beta_{j1}, \ldots, \beta_{jk}, \gamma_{ji}, \delta_{ji} \in \mathbb{N}$ *for* $1 \le i < j$;

(iii) $P = c X_1^{\zeta_1} \cdots X_k^{\zeta_k} \prod_{1 \le j \le a} Q_j^{\eta_j}$,
where $c \in \mathbb{R}$, *and* $\zeta_1, \ldots, \zeta_k, \eta_1, \ldots, \eta_a \in \mathbb{Z}$.

Remark 3.2. Observe that in Lemma 3.1 all exponents other than those in line (iii) are in $\mathbb{N}$ rather than in $\mathbb{Z}$ (cf. Definition 1.6). Observe also that if a polynomial P satisfies the conditions of the lemma, then it has additive complexity at most a.

In the proof of Lemma 3.1, we are going to use the following notation.

Notation 3.3. For $a \in \mathbb{R}$ we let

$$a^+ = \begin{cases} a & \text{if } a \ge 0 \\ 0 & \text{if } a \le 0 \end{cases} \qquad a^- = \begin{cases} 0 & \text{if } a \ge 0 \\ -a & \text{if } a \le 0. \end{cases}$$

For a sequence $\alpha = (a_1, \ldots, a_s) \in \mathbb{R}^s$ we set $\alpha^+ = (a_1^+, \ldots, a_s^+)$ and $\alpha^- = (a_1^-, \ldots, a_s^-)$ and notice that $\alpha = \alpha^+ - \alpha^-$ and that all the entries of both α^+ and α^- are non-negative.

Proof of Lemma 3.1. The "if" part is clear. We prove below the "only if" part.

Suppose that P has additive complexity bounded by a. Let $\mathcal{P}(P; a, n)$ be the property that P has additive complexity at most a, and that there exists a additive representation of P (see Definition 1.6) in which all exponents in the first $n-1$ equations are non-negative. Notice that Property $\mathcal{P}(P; a, 1)$ holds, since it only asserts that P has additive complexity at most a, which we have assumed.

We now prove that $\mathcal{P}(P; a, n)$ implies $\mathcal{P}(P; a, n+1)$. Suppose that $\mathcal{P}(P; a, n)$ holds. Hence, there is a additive representation of P of the form (*) such that no negative exponents appear in the first $n-1$ equations. We now define $\widetilde{Q}_n$ as follows. Suppose that

$$Q_n = \underbrace{\overbrace{\frac{u_n \mathbf{X}^{\alpha_n^+} Q_1^{\gamma_{n,1}^+} \cdots Q_{n-1}^{\gamma_{n,n-1}^+}}{\mathbf{X}^{\alpha_n^-} Q_1^{\gamma_{n,1}^-} \cdots Q_{n-1}^{\gamma_{n,n-1}^-}}}^{f_1}}_{g_1} + \underbrace{\overbrace{\frac{v_n \mathbf{X}^{\beta_n^+} Q_1^{\delta_{n,1}^+} \cdots Q_{n-1}^{\delta_{n,n-1}^+}}{\mathbf{X}^{\beta_n^-} Q_1^{\delta_{n,1}^-} \cdots Q_{n-1}^{\delta_{n,n-1}^-}}}^{f_2}}_{g_2} = \frac{f_1 g_2 + f_2 g_1}{g_1 g_2}.$$

Define

$$\widetilde{Q}_n = f_1 g_2 + f_2 g_1.$$

Observe that

$$\begin{aligned} Q_{n+1} = \quad & u_{n+1} \mathbf{X}^{\alpha_{n+1}} \prod_{0 \le i \le n} Q_i^{\gamma(n+1,i)} \quad + \quad v_{n+1} \mathbf{X}^{\beta_{n+1}} \prod_{0 \le i \le n} Q_i^{\delta(n+1,i)} \\ = \quad & u_\cdot \mathbf{X}^{\alpha_\cdot} \prod_{0 \le i \le n-1} Q_i^{\gamma(\cdot,i)} \cdot \left(\frac{\widetilde{Q}_n}{g_1 g_2} \right)^{\gamma(\cdot,n)} + v_\cdot \mathbf{X}^{\beta_\cdot} \prod_{0 \le i \le n-1} Q_i^{\delta(\cdot,i)} \cdot \left(\frac{\widetilde{Q}_n}{g_1 g_2} \right)^{\delta(\cdot,n)} \\ = \; & u_\cdot \mathbf{X}^{\alpha_\cdot - (\alpha_n^- + \beta_n^-)} \prod_{0 \le i \le n-1} Q_i^{\gamma(\cdot,i) - (\gamma^-_{(n,i)} + \delta^-_{(n,i)})(\gamma(\cdot,n))} \cdot \widetilde{Q}_n^{\gamma(\cdot,n)} \\ & + v_\cdot \mathbf{X}^{\beta_\cdot - (\alpha_n^- + \beta_n^-)} \prod_{0 \le i \le n-1} Q_i^{\delta(\cdot,i) - (\gamma^-_{(n,i)} + \delta^-_{(n,i)})(\delta(\cdot,n))} \cdot \widetilde{Q}_n^{\delta(\cdot,n)} \end{aligned}$$

(where each "$n+1$" in the subscript/superscript from the second line down has been replaced with $\cdot$ in order to fit inside the page width).

Thus, by replacing Q_n with $\widetilde{Q_n}(g_1g_2)^{-1}$ in the additive sequence representing P, we obtain a new sequence in which all exponents in the first n equations are non-negative, which proves $\mathcal{P}(P; a, n+1)$.

□

3.2. **The algebraic case.** Before proving Theorem 1.9 it is useful to first consider the algebraic case separately, since the main technical ingredients used in the proof of Theorem 1.9 are more clearly visible in this case. With this in mind, in this section we consider the algebraic case and prove the following theorem, deferring the proof in the general semi-algebraic case till the next section.

Theorem 3.4. *The number of semi-algebraic homotopy types of* $\mathrm{Zer}(F, \mathbb{R}^k)$ *amongst all polynomials* $F \in \mathbb{R}[X_1, \ldots, X_k]$ *having additive complexity at most* a *does not exceed*

$$2^{(ka)^{O(1)}}.$$

Before proving Theorem 3.4 we need a few preliminary results.

Proposition 3.5. *Let* $F, P, Q \in \mathbb{R}[\mathbf{X}]$ *with* $F = \frac{P}{Q} \in \mathbb{R}[\mathbf{X}]$*,* $R \in \mathbb{R}, R > 0$*,*

$$T = \{(\boldsymbol{x}, t) \in \mathbb{R}^{k+1} | \ (P^2(\boldsymbol{x}) \leq t(Q^2(\boldsymbol{x}) - t^N) \wedge (|\boldsymbol{x}|^2 \leq R^2))\},$$

where $N = 2\deg(Q) + 1$*, and*

$$T_0 := \pi_{[1,k]}(\overline{T} \cap \pi_{k+1}^{-1}(0)).$$

Then,

$$T_0 = \mathrm{Zer}(F, \mathbb{R}^k) \cap \overline{B_k(0, R)}.$$

Before proving Proposition 3.5 we first discuss an illustrative example.

Example 3.6. Let

$$F_1 = X(X^2 + Y^2 - 1),$$
$$F_2 = X^2 + Y^2 - 1.$$

Also, let

$$P_1 = X^2(X^2 + Y^2 - 1),$$
$$P_2 = X(X^2 + Y^2 - 1),$$

and

$$Q_1 = Q_2 = X.$$

Then as rational functions in X, Y we have that

$$F_1 = \frac{P_1}{Q_1},$$

and

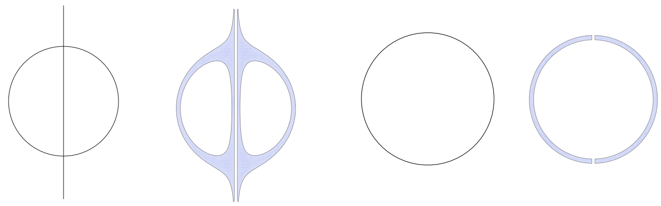
$$F_2 = \frac{P_2}{Q_2}.$$

For $i = 1, 2$, and $R > 0$, let

$$T_i = \{(\mathbf{x}, t) \in \mathbb{R}^{k+1} | \ (P_i^2(\mathbf{x}) \leq t(Q_i^2(\mathbf{x}) - t^N) \wedge (|\mathbf{x}|^2 \leq (R+t)^2))\}$$

as in Proposition 3.5.

In Figure 1, we display from left to right, $\mathrm{Zer}(F_1,\mathbb{R}^2)$, $(T_1)_\varepsilon$, $\mathrm{Zer}(F_2,\mathbb{R}^2)$ and and $(T_2)_\varepsilon$, respectively (where $\varepsilon = .005$ and $N = 3$). Notice that for $i = 1,2$, and any fixed $R > 0$, the semi-algebraic set $(T_i)_\varepsilon$ approaches (in the sense of Hausdorff limit) the set $\mathrm{Z}(F_i,\mathbb{R}^2) \cap \overline{B_2(0,R)}$ as $\varepsilon \to 0$.



(a) (b) (c) (d)

Figure 1: Two examples.

We now prove Proposition 3.5.

Proof of Proposition 3.5. We show both inclusions. First let $\mathbf{x} \in T_0$, and we show that $F(\mathbf{x}) = 0$. For every $\varepsilon > 0$ we prove that $0 \le F^2(\mathbf{x}) < \varepsilon$ which suffices to prove this inclusion. Since F is continuous there exists $\delta > 0$ such that

$$|\mathbf{x}-\mathbf{y}|^2 < \delta \quad \Longrightarrow \quad |F^2(\mathbf{x}) - F^2(\mathbf{y})| < \tfrac{\varepsilon}{2}. \tag{3.1}$$

After possibly making δ smaller we can suppose that $\delta < \frac{\varepsilon^2}{4}$.

¿From the definition of T_0 we have that

$$T_0 = \{\mathbf{x} \mid \quad (\forall\delta)(\delta > 0 \implies (\exists t)(\exists \mathbf{y})(\mathbf{y} \in T_t \wedge |\mathbf{x}-\mathbf{y}|^2 + t^2 < \delta))\}. \tag{3.2}$$

Since $\mathbf{x} \in T_0$ we can conclude that there exists $t > 0$ such that there exists $\mathbf{y} \in T_t$ such that $|\mathbf{x}-\mathbf{y}|^2+t^2 < \delta$, and in particular both $|\mathbf{x}-\mathbf{y}|^2 < \delta$ and $t^2 < \delta < \frac{\varepsilon^2}{4}$. The former inequality implies that $|F^2(\mathbf{x}) - F^2(\mathbf{y})| < \frac{\varepsilon}{2}$. The latter inequality implies $t < \frac{\varepsilon}{2}$, and together with $\mathbf{y} \in T_t$ we have the following implications.

$$\begin{aligned} & P^2(\mathbf{y}) \le t(Q^2(\mathbf{y}) - t^N) \\ \Longrightarrow\ & Q^2(\mathbf{y})F^2(\mathbf{y}) \le t(Q^2(\mathbf{y}) - t^N) \\ \Longrightarrow\ & 0 \le F^2(\mathbf{y}) \le t - \frac{t^{N+1}}{Q^2(\mathbf{y})} < t \\ \Longrightarrow\ & 0 \le F^2(\mathbf{y}) < \tfrac{\varepsilon}{2} \end{aligned}$$

So, $F^2(\mathbf{y}) < \frac{\varepsilon}{2}$. Finally, note that $|F^2(\mathbf{x})| \le |F^2(\mathbf{x}) - F^2(\mathbf{y})| + |F^2(\mathbf{y})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

We next prove the other inclusion, namely we show $\mathrm{Zer}(F,\mathbb{R}^k) \cap \overline{B_k(0,R)} \subseteq T_0$. Let $\mathbf{x} \in \mathrm{Zer}(F,\mathbb{R}^k) \cap \overline{B_k(0,R)}$. We fix $\delta > 0$ and show that there exists $t \in \mathbb{R}$ and $\mathbf{y} \in T_t$ such that $|\mathbf{x}-\mathbf{y}| + t^2 < \delta$ (cf. Equation 3.2).

There are two cases to consider.

$Q(\mathbf{x}) \ne 0$: Since $Q(\mathbf{x}) \ne 0$, there exists $t > 0$ such that $Q^2(\mathbf{x}) \ge t^N$ and $t^2 < \delta$. Now, $\mathbf{x} \in T_t$ and

$$|\mathbf{x}-\mathbf{x}| + t^2 \;=\; t^2 \;< \delta,$$

so setting $\mathbf{y} = \mathbf{x}$ we see that $\mathbf{y} \in T_t$ and $|\mathbf{x}-\mathbf{y}| + t^2 < \delta$, and hence $\mathbf{x} \in T_0$.

$Q(\mathbf{x}) = 0$: Let $\mathbf{v} \in \mathbb{R}^k$ be generic, and denote $\widehat{P}(U) = P(\mathbf{x}+U\mathbf{v})$, $\widehat{Q}(U) = Q(\mathbf{x}+U\mathbf{v})$, and $\widehat{F}(U) = F(\mathbf{x}+U\mathbf{v})$. Note that

$$\begin{aligned} \widehat{P} &= \widehat{Q}\widehat{F}, \\ \widehat{P}(0) &= \widehat{Q}(0) = \widehat{F}(0) = 0. \end{aligned} \tag{3.3}$$

Moreover, if F is not the zero polynomial, we claim that for a generic $\mathbf{v} \in \mathbb{R}^k$, $\widehat{P}$ is not identically zero. Assume F is not identically zero, and hence P is not identically zero. In order to prove that $\widehat{P}$ is not identically zero for a generic choice of $\mathbf{v}$, write $P = \sum_{0 \leq i \leq d} P_i$ where P_i is the homogeneous part of P of degree i, and P_d not identically zero. Then, it is easy to see that $\widehat{P}(U) = P_d(\mathbf{v})U^d +$ lower degree terms. Since $\mathbb{R}$ is an infinite field, a generic choice of $\mathbf{v}$ will avoid the set of zeros of P_d and $\widehat{P}$ is then not identically zero. Furthermore, we require that $\mathbf{x} + t\mathbf{v} \in B_k(0,R)$ for $t > 0$ sufficiently small. For generic $\mathbf{v}$, this is true for either $\mathbf{v}$ or $-\mathbf{v}$, and so after possibly replacing $\mathbf{v}$ by $-\mathbf{v}$ (and noticing that since P_d is homogeneous we have $P_d(\mathbf{v}) = (-1)^d P_d(-\mathbf{v})$) we may assume $\mathbf{x} + t\mathbf{v} \in B_k(0,R)$ for $t > 0$ sufficiently small. Denoting by $\nu = \mathrm{mult}_0(\widehat{P})$ and $\mu = \mathrm{mult}_0(\widehat{Q})$, we have from (3.3) that $\nu > \mu$.

Let

$$\begin{aligned} \widehat{P}(U) &= \sum_{i=\nu}^{\deg_U \widehat{P}} c_i U^i = U^\nu \cdot \sum_{i=0}^{\deg_U \widehat{P}-\nu} c_{\nu+i} U^i = c_\nu U^\nu + \text{ (higher order terms)}, \\ \widehat{Q}(U) &= \sum_{i=\mu}^{\deg_U \widehat{Q}} d_i U^i = U^\mu \cdot \sum_{i=0}^{\deg_U \widehat{Q}-\mu} d_{\mu+i} U^i = d_\mu U^\mu + \text{ (higher order terms)} \end{aligned}$$

where $c_\nu, d_\mu \neq 0$.

Then we have

$$\begin{aligned} \widehat{P}^2(U) &= c_\nu^2 U^{2\nu} + \text{ (higher order terms)}, \\ \widehat{Q}^2(U) &= d_\mu^2 U^{2\mu} + \text{ (higher order terms)}, \\ D(U) := U(\widehat{Q}^2(U) - U^N) &= U(d_\mu^2 U^{2\mu} + \text{ (higher order terms)} - U^N), \\ D(U) - \widehat{P}^2(U) &= d_\mu^2 U^{2\mu+1} + \text{ (higher order terms)} - U^{N+1}. \end{aligned}$$

Since $\mu \leq \deg(Q)$ and $N = 2\deg(Q) + 1$, we have that $2\mu + 1 < N + 1$. Hence, there exists $t_1 > 0$ such that for t with $0 < t < t_1$, we have that $D(t) - \widehat{P}^2(t) \geq 0$, and thus $\mathbf{x} + t\mathbf{v} \in T_t$. Let $t_2 = (\frac{\delta}{|\mathbf{v}|^2+1})^{1/2}$ and note that for all $t \in \mathbb{R}$, $0 < t < t_2$, we have $(|\mathbf{v}|^2+1)t^2 < \delta$. Since $\mathbf{x} \in \overline{B_k(0,R)}$ and by our choice of $\mathbf{v}$, there exists t_3 such that for all $0 < t < t_3$ we have $\mathbf{x} + t\mathbf{v} \in \mathrm{Reali}(|\mathbf{X}|^2 \leq R^2)$. Finally, if $t > 0$ satisfies $0 < t < \min\{t_1, t_2, t_3\}$ then $\mathbf{x} + t\mathbf{v} \in T_t$, and

$$|\mathbf{x} - (\mathbf{x} + t\mathbf{v})|^2 + t^2 \ = \ (|\mathbf{v}|^2+1)t^2 < \delta,$$

and so we have shown that $\mathbf{x} \in T_0$.

The case where F is the zero polynomial is straightforward.

□

Proof of Theorem 3.4. If F has additive complexity at most a, then there exists by Lemma 3.1, $P, Q \in \mathbb{R}[X_1, \ldots, X_k]$ such that $F = \frac{P}{Q}$, with P, Q each having division-free additive complexity at most a. Hence, $P^2 - t(Q^2 - t^N) \in \mathbb{R}[X_1, \ldots, X_k, t]$ has division-free additive complexity bounded by $2a + 1$. Let $R > 0$ and let

$$T = \{(\mathbf{x}, t) \in \mathbb{R}^{k+1} | \ (P^2(\mathbf{x}) \le t(Q^2(\mathbf{x}) - t^N) \wedge (|\mathbf{x}|^2 \le R^2 + 1))\}$$

and

$$T_0 := \pi_{[1,k]}(\overline{T} \cap \pi_{k+1}^{-1}(0)).$$

By Proposition 3.5 we have that

$$T_0 = \mathrm{Zer}(F, \mathbb{R}^k) \cap \overline{B_k(0, R)}.$$

By the conical structure at infinity of semi-algebraic sets (see for instance [4, pg. 188]) we have that for all sufficiently large $R > 0$, the semi-algebraic sets $\mathrm{Zer}(F, \mathbb{R}^k) \cap \overline{B_k(0, R)}$ and $\mathrm{Zer}(F, \mathbb{R}^k)$ are semi-algebraically homeomorphic.

Note the one-parameter semi-algebraic family T (where the last co-ordinate is the parameter) is described by a formula having division-free additive format $(2a + k + 1, k + 1)$.

By Theorem 2.1 applied to T we obtain a collection of semi-algebraic sets $\mathcal{S}_{2a+k+1,k}$ such that T_0, and hence $\mathrm{Zer}(F, \mathbb{R}^k)$, is homotopy equivalent to some $S \in \mathcal{S}_{2a+k+1,k}$ and $\#\mathcal{S} = 2^{((2a+k+1)k)^{O(1)}} = 2^{(ka)^{O(1)}}$, which proves the theorem.

□

3.3. **The semi-algebraic case.** We first prove a generalization of Proposition 3.5.

Notation 3.7. Let $\mathbf{X} = (X_1, \ldots, X_k)$ be a block of variables and $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ with $\sum_{i=1}^n k_i = k$. Let $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$ with $r_i > 0$, $i = 1, \ldots, n$. Let $B_{\mathbf{k}}(0, \mathbf{r})$ denote the product

$$B_{\mathbf{k}}(0, \mathbf{r}) := B_{k_1}(0, r_1) \times \cdots \times B_{k_n}(0, r_n).$$

We will also use the following notation (already used in Proposition 3.5)

Notation 3.8. Given any one-parameter semi-algebraic family $T \subset \mathbb{R}^{k+1}$ (parameterized by the last co-ordinate) we will denote by

$$T_0 := \pi_{[1,k]}(\overline{T} \cap \pi_{k+1}^{-1}(0)).$$

Proposition 3.9. *Let* $F_1, \ldots, F_s, P_1, \ldots, P_s, Q_1, \ldots, Q_s \in \mathbb{R}[\mathbf{X}^1, \ldots, \mathbf{X}^n]$, $\mathcal{P} = \{F_1, \ldots, F_s\}$, *with* $F_i = \frac{P_i}{Q_i} \in \mathbb{R}[\mathbf{X}^1, \ldots, \mathbf{X}^n]$. *Suppose* $\mathbf{X}^i = (X_1^i, \ldots, X_{k_i}^i)$ *and let* $\mathbf{k} = (k_1, \ldots, k_n)$. *Suppose* ϕ *is a* $\mathcal{P}$*-formula containing no negations and no inequalities. Let*

$$\bar{P}_i := P_i \prod_{j \neq i} Q_j,$$

$$\bar{Q} := \prod_j Q_j,$$

and let $\bar{\phi}$ *denote the formula by replacing each* $F_i = 0$ *with*

$$\bar{P}_i^2 \le U(\bar{Q}^2 - U^N),$$

$N = 2\deg(\bar{Q}) + 1$. *Then, for every* $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$ *such that* $r_i > 0$ *for all* i, *we have*

$$\mathrm{Reali}\left(\bigwedge_{i=1}^{n}(|\mathbf{X}^i|^2 \le r_i^2) \wedge \bar{\phi}\right)_0 = \mathrm{Reali}(\phi) \cap \overline{B_{\mathbf{k}}(0,\mathbf{r})}.$$

(See Notation 3.8 for the definition of $\mathrm{Reali}\left(\bigwedge_{i=1}^{n}(|\mathbf{X}^i|^2 \le r_i^2) \wedge \bar{\phi}\right)_0$).

Proof. We follow the proof of Proposition 3.5. The only case which is not immediate is the case $\mathbf{x} \in \mathrm{Reali}(\phi) \cap \overline{B_{\mathbf{k}}(0,\mathbf{r})}$ and $\bar{Q}(\mathbf{x}) = 0$.

Suppose $\mathbf{x} \in \mathrm{Reali}(\phi) \cap \overline{B_{\mathbf{k}}(0,\mathbf{r})}$ and that $\bar{Q}(\mathbf{x}) = 0$. Since ϕ is a formula containing no negations and no inequalities, it consists of conjunctions and disjunctions of equalities. Without loss of generality we can assume that ϕ is written as a disjunction of conjunctions, and still without negations. Let

$$\phi = \bigvee_{\alpha} \phi_\alpha$$

where ϕ_α is a conjunction of equations. As above let $\bar{\phi}_\alpha$ be the formula obtained from ϕ_α after replacing each $F_i = 0$ in ϕ_α with

$$\bar{P_i}^2 \le U(\bar{Q}^2 - U^N),$$

$N = 2\deg(\bar{Q}) + 1$.

We have

$$\begin{aligned}
\mathrm{Reali}\left(\bigwedge_{i=1}^{p}(|\mathbf{X}^i|^2 \le r_i^2) \wedge \bar{\phi}\right)_0 &= \mathrm{Reali}\left(\bigwedge_{i=1}^{p}(|\mathbf{X}^i|^2 \le r_i^2) \wedge \left(\bigvee_{\alpha} \bar{\phi}_\alpha\right)\right)_0 \\
&= \mathrm{Reali}\left(\bigvee_{\alpha}\bigwedge_{i=1}^{p}(|\mathbf{X}^i|^2 \le r_i^2) \wedge \bar{\phi}_\alpha\right)_0 \\
&= \bigcup_{\alpha}\mathrm{Reali}\left(\bigwedge_{i=1}^{p}(|\mathbf{X}^i|^2 \le r_i^2) \wedge \bar{\phi}_\alpha\right)_0
\end{aligned}$$

since $(T \cup S)_0 = T_0 \cup S_0$. In order to show that $\mathbf{x} \in \mathrm{Reali}\left(\bigwedge_{i=1}^{p}(|\mathbf{X}^i|^2 \le r_i^2) \wedge \bar{\phi}\right)_0$ it now suffices to show that if $\mathbf{x} \in \mathrm{Reali}(\phi_\alpha) \cap \overline{B_{\mathbf{k}}(0,\mathbf{r})}$ and $\bar{Q}(\mathbf{x}) = 0$, then $\mathbf{x} \in \mathrm{Reali}\left(\bigwedge_{i=1}^{p}(|\mathbf{X}^i|^2 \le r_i^2) \wedge \bar{\phi}_\alpha\right)_0$.

Let $\mathbf{x} \in \mathrm{Reali}(\phi_\alpha) \cap \overline{B_{\mathbf{k}}(0,\mathbf{r})}$ and suppose $\bar{Q}(\mathbf{x}) = 0$. Let $\mathcal{Q} \subseteq \mathcal{P}$ consist of the polynomials of $\mathcal{P}$ appearing in ϕ_α. Let $\mathbf{v} \in \mathbb{R}^k$ be generic, and denote $\widehat{P}_i(U) = \bar{P}_i(\mathbf{x} + U\mathbf{v})$, $\widehat{Q}(U) = \bar{Q}(\mathbf{x} + U\mathbf{v})$, and $\widehat{F}_i(U) = \bar{F}(\mathbf{x} + U\mathbf{v})$. Note that

$$\widehat{P}_i = \widehat{Q}\widehat{F}_i,$$

$$\widehat{P}_i(0) = \widehat{Q}(0) = \widehat{F}_i(0) = 0. \tag{3.4}$$

As in the proof of Proposition 3.5, if $F_i \in \mathcal{Q}$ is not the zero polynomial then for generic $\mathbf{v} \in \mathbb{R}^k$, $\widehat{P}_i$ is not identically zero. Since ϕ_α consists of a conjunction of equalities and

$$\bigwedge_{\substack{F \in \mathcal{Q} \\ F \not\equiv 0}} F = 0 \iff \bigwedge_{F \in \mathcal{Q}} F = 0,$$

we may assume that $\mathcal{Q}$ does not contain the zero polynomial. Under this assumption, and for a generic $\mathbf{v} \in \mathbb{R}^k$, we have that for every $F_i \in \mathcal{Q}$ the univariate polynomial $\widehat{P}_i$ is not identically zero. As in the proof of Proposition 3.5, we can assume that $\mathbf{v}$ also satisfies $\mathbf{x} + t\mathbf{v} \in B_{\mathbf{k}}(0, \mathbf{r})$ for $t > 0$ sufficiently small. Denoting by $\nu_i = \mathrm{mult}_0(\widehat{P}_i)$ and $\mu = \mathrm{mult}_0(\widehat{Q})$, we have from (3.3) that $\nu_i > \mu$ for all $i = 1, \ldots, s$.

Let

$$\widehat{P}_i(U) = \sum_{j=\nu_i}^{\deg_U \widehat{P_i}} c_j U^j = U^{\nu_i} \cdot \sum_{j=0}^{\deg_U \widehat{P_i} - \nu_i} c_{\nu_i + j} U^j = c_{\nu_i} U^{\nu_i} + \text{ (higher order terms)},$$

$$\widehat{Q}(U) = \sum_{j=\mu}^{\deg_U \widehat{Q}} d_j^j = U^{\mu} \cdot \sum_{j=0}^{\deg_U \widehat{Q} - \mu} d_{\mu+j} U^j = d_\mu U^\mu + \text{ (higher order terms)}$$

where $d_\mu \neq 0$ and $c_{\nu_i} \neq 0$.

Then we have

$$\widehat{P}_i^{\,2}(U) = c_{\nu_i}^2 U^{2\nu_i} + \text{ (higher order terms)},$$

$$\widehat{Q}^2(U) = d_\mu^2 U^{2\mu} + \text{ (higher order terms)},$$

$$D(t) := U(\widehat{Q}^2(U) - U^N) = U(d_\mu^2 U^{2\mu} + \text{ (higher order terms)} - U^N),$$

$$D(U) - \widehat{P}_i^{\,2}(U) = d_\mu^2 U^{2\mu+1} + \text{ (higher order terms)} - U^{N+1}.$$

Since $\mu \leq \deg(\bar{Q})$ and $N = 2\deg(\bar{Q}) + 1$, we have that $2\mu + 1 < N + 1$. Hence, there exists $t_{1,i} > 0$ such that for t with $0 < t < t_{1,i}$, we have that $D(t) - \widehat{P}_i^{\,2}(t) \geq 0$, and thus $\mathbf{x} + t\mathbf{v}$ satisfies

$$\bar{P}_i^2(\mathbf{x} + t\mathbf{v}) \leq t(\bar{Q}^2(\mathbf{x} + t\mathbf{v}) - t^N).$$

Let $t_1 = \min\{t_{1,1}, \ldots, t_{1,s}\}$. Let $t_2 = (\frac{\delta}{|\mathbf{v}|^2+1})^{1/2}$ and note that for all $t \in \mathbb{R}$, $0 < t < t_2$, we have $(|\mathbf{v}|^2 + 1)t^2 < \delta$. Since $\mathbf{x} \in \overline{B_{\mathbf{k}}(0, \mathbf{r})}$ and by our choice of $\mathbf{v}$, there exists t_3 such that for all $0 < t < t_3$ we have

$$\mathbf{x} + t\mathbf{v} \in \mathrm{Reali}\left(\bigwedge_{i=1}^{p} (|\mathbf{X}^i|^2 \leq r_i^2)\right).$$

Finally, if $t > 0$ satisfies $0 < t < \min\{t_1, t_2, t_3\}$ then

$$(\mathbf{x} + t\mathbf{v}, t) \in \mathrm{Reali}\left(\bigwedge_{i=1}^{p} (|\mathbf{X}^i|^2 \leq r_i^2) \wedge \bar{\phi}_\alpha\right)$$

and

$$|\mathbf{x} - (\mathbf{x} + t\mathbf{v})|^2 + t^2 \;=\; (|\mathbf{v}|^2)t^2 < \delta,$$

and so we have shown that

$$\mathbf{x} \in \mathrm{Reali}\left(\bigwedge_{i=1}^{p} (|\mathbf{X}^i|^2 \leq r_i^2) \wedge \bar{\phi}_\alpha\right)_0.$$

□

Definition 3.10. Let Φ be a $\mathcal{P}$-formula, $\mathcal{P} \subseteq \mathbb{R}[\mathbf{X}_1, \ldots, \mathbf{X}_k]$, and say that Φ is a $\mathcal{P}$-*closed formula* if the formula Φ contains no negations and all the inequalities in atoms of Φ are weak inequalities.

Let $\mathcal{P} = \{F_1, \ldots, F_s\} \subset \mathbb{R}[X_1, \ldots, X_k]$, and Φ a $\mathcal{P}$-closed formula.

For $R > 0$, let Φ_R denote the formula $\Phi \wedge (|\mathbf{X}|^2 - R^2 \leq 0)$.

Let $\Phi^\dagger$ be the formula obtained by replacing each occurrence of the atom $F_i * 0$, $* \in \{=, \leq, \geq\}$, $i = 1, \ldots, s$, with

$$\begin{aligned} F_i - T_i^2 &= 0 \qquad * \in \{\leq\}, \\ -F_i - T_i^2 &= 0 \qquad * \in \{\geq\}, \\ F_i &= 0 \qquad * \in \{=\}, \end{aligned}$$

and let for $R, R' > 0$, let $\Phi^\dagger_{R,R'}$ denote the formula $\Phi^\dagger \wedge (U_1^2 + |\mathbf{X}|^2 - R^2 = 0) \wedge (U_2^2 + |\mathrm{T}|^2 - R'^2 = 0)$.

We have

Proposition 3.11.

$$\mathrm{Reali}(\Phi) = \pi_{[1,k]}(\mathrm{Reali}(\Phi^\dagger)),$$

and for all $0 < R \ll R'$,

$$\mathrm{Reali}(\Phi_R) = \pi_{[1,k]}(\mathrm{Reali}(\Phi^\dagger_{R,R'})),$$

Proof. Obvious.

□

Notice that the formula $\Phi^\dagger_{R,R'}$ in Proposition 3.11 has no negations, and only equalities and no weak inequalities.

In what follows we fix Φ and $R, R' > 0$. Let

$$S_{R,R'} = \mathrm{Reali}(\Phi^\dagger_{R,R'}).$$

Note that for $0 < R \ll R'$, $\pi_{[1,k]}|_{S_{R,R'}}$ is a continuous, semi-algebraic surjection onto $\mathrm{Reali}(\Phi_R)$.

Let $\pi_{R,R'}$ denote the map $\pi_{[1,k]}|_{S_{R,R'}}$.

Proposition 3.12. *We have that* $\mathcal{J}^p_{\pi_{R,R'}}(S_{R,R'})$ *is p-equivalent to* $\pi_{[1,k]}(S_{R,R'})$. *Moreover, for any two formulas* Φ, Ψ, *the realizations* $\mathrm{Reali}(\Phi)$ *and* $\mathrm{Reali}(\Psi)$ *are homotopy equivalent if, for all* $1 \ll R \ll R'$, $\mathrm{Reali}(\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})), \mathrm{Reali}(\mathcal{J}^p_{\pi_{R,R'}}(\Psi^\dagger_{R,R'}))$ *are homotopy equivalent for some* $p > k$.

Proof. Immediate from Proposition 2.12 and Propositions 2.4 and 3.11.

□

Proposition 3.13. *Suppose that the sum of the additive complexities of* $F_i, 1 \leq i \leq s$ *is bounded by* a. *Then the semi-algebraic set* $\mathrm{Reali}\left(\overline{\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})}\right)$ *(cf. Proposition 3.9) can be defined by a* $\mathcal{P}'$*-formula with* $\mathcal{P}' \in \mathcal{A}_{3M^2,N+1}$,

$$\begin{aligned} M &= (p+1)(k+3a+8) + 2\binom{p+1}{2}(k+a+4), \\ N &= (p+1)(k+a+3) + \binom{p+1}{2}. \end{aligned}$$

Proof. If Φ is a $\mathcal{P}$ formula, $\mathcal{P} = \{F_1, \ldots, F_s\}$, such that then $\Phi^\dagger$ has additive format bounded by (a, k), then $\Phi^\dagger$ has additive format bounded by $(2a, k + a)$. From the definition of $\Phi^\dagger_{R,R'}$, it is clear that if $\Phi^\dagger$ has additive format bounded by $(2a, k+a)$, then $\Phi^\dagger_{R,R'}$ has additive format bounded by $(2a+6, k+a+2)$. From the definition of $\mathcal{J}^p_f(\Phi)$ (cf. Definition 2.10 and Equation 2.1), in the case where $f = \pi_{R,R'}$ and the formula $\Phi^\dagger_{R,R'}$, we have that the additive format of $\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})$ is bounded by (M, N),

$$M = (p+1)(k + 3a + 8) + 2\binom{p+1}{2}(k + a + 4),$$

$$N = (p+1)(k + a + 3) + \binom{p+1}{2}.$$

In the above, for $f = \pi_{R,R'}$, the estimates of Proposition 2.18 suffice, with (a, k) replaced by $(2a+6, k+a+2)$. Finally, if ϕ has additive format bounded by (a, k), then $\bar{\phi}$ (cf. Proposition 3.9) has additive format bounded by $(2a^2+a, k+1)$. Thus, we have $\overline{\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})}$ has additive format bounded by $(2M^2 + M, N + 1)$. After making the estimate $2M^2 + M \leq 3M^2$ the proposition follows.

□

Finally, we obtain

Proposition 3.14. *The number of distinct homotopy types of bounded semi-algebraic subsets of defined by $\mathcal{P}$-closed formulas with $\mathcal{P} \in \mathcal{A}_{a,k}$ is bounded by $2^{(ka)^{O(1)}}$.*

Proof. By the conical structure at infinity of semi-algebraic sets (see for instance [4, pg. 188]) we have that for all sufficiently large $R > 0$, the sets $\mathrm{Reali}(\Phi_R) \approx \mathrm{Reali}(\Phi)$ are semi-algebraically homeomorphic.

By Proposition 3.12 it suffices to bound the number of possible homotopy types of the set $\mathcal{J}^p_{\pi_{R,R'}}(S_{R,R'})$, $0 \ll R \ll R'$. The proposition now follows from Propositions 3.9 (that $\left(\mathrm{Reali}\left(\overline{\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})}\right)\right)_0 = \mathrm{Reali}(\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'}))$) and 3.13 (for bounding the additive complexity of the formula $\overline{\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})}$) and Theorem 2.1 (for bounding the number of possible homotopy types of the limit $\left(\mathrm{Reali}\left(\overline{\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})}\right)\right)_0$) applied to $\mathrm{Reali}\left(\overline{\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})}\right)$. □

Remark 3.15. It is important in the proof of the above proposition that the formula $\mathcal{J}^p_{\pi_{R,R'}}(\Phi^\dagger_{R,R'})$ is of the form $\Omega^R \wedge \Theta_1 \wedge \Theta_2^{\Phi^\dagger_{R,R'}} \wedge \Theta_3^{\pi_{R,R'}}$. In particular, $\Theta_1 \wedge \Theta_2^{\Phi^\dagger_{R,R'}} \wedge \Theta_3^{\pi_{R,R'}}$ contains no negations or inequalities.

Proof of Theorem 1.9. Using the construction of Gabrielov and Vorobjov [10] one can reduce the case of arbitrary semi-algebraic sets to that of closed and bounded one, defined by a $\mathcal{P}$-closed formula, without changing asymptotically the complexity estimates (see for example [5]). The theorem then follows directly from Proposition 3.14 above.

□

3.4. **Proof of Theorem 1.12.**

Proof of Theorem 1.12. The proof is identical to that of the proof of Theorem 2.1, except that we use Theorem 1.9 instead of Theorem 1.5. □

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