

Def: u is called a unit vector of \mathbb{R}^n if $\|u\| = 1$
 Ex: $u \neq 0$ $\frac{u}{\|u\|}$ is a unit vector.

Properties of standard inner product.

- (a) $u \cdot u \geq 0$ $u \cdot u = 0$ iff $u = 0$
- (b) $u \cdot v = v \cdot u$
- (c) $(u+v) \cdot w = u \cdot w + v \cdot w$
- (d) $(cu) \cdot v = c(u \cdot v)$ $c \in \mathbb{R}$

For any V.S. V , an inner product (I.P) is a map: $(u, v) \rightarrow \mathbb{R}$ satisfying (a) - (d)

Ex: $(u, v) := u \cdot v$ the dot product is a inner product.

Remark: (i) $(u, v+w) = (v+w, u) = (v, u) + (w, u) = (u, v) + (u, w)$
 (ii) (d) $\Rightarrow (u, cv) = c(u, v) = \underline{(cu, v)}$

Examples in \mathbb{R}^n

Ex: $\{v_1, \dots, v_n\}$ a basis for \mathbb{R}^n
 $u = \sum a_i v_i$ $w = \sum b_i v_i$ $(u, w) := \sum a_i b_i$ is a I.P.

Proof: (b) - (d) is clearly $u \cdot u = \sum a_i^2 \geq 0$
 $u \cdot u = 0 \Rightarrow a_i^2 = 0 \Rightarrow a_i = 0 \Rightarrow u = \sum 0 \cdot v_i = 0$

How to interpret a non-standard inner product?

standard I.P \rightarrow a measure for length. \rightarrow different in north pole / equator.

(...) I.P $S = \{v_1, \dots, v_n\}$ a basis $u = \sum a_i v_i$ $w = \sum b_i v_i$

$$(u, w) = [u]_S^T C [w]_S$$

$[u]_S$ is the coordinates w.r.t. S

$$C = [c_{ij}] \quad c_{ij} = (v_i, v_j)$$

(Post the proof as an example)

(b) $\Rightarrow C$ sym } C positive definite
 (a) $\Rightarrow x^T C x > 0$ $x \neq 0$ }

Def: C is called the matrix representation of (...) w.r.t. S .

Post Proof

Theorem: (\cdot, \cdot) is a I.P. in \mathbb{R}^n , iff $(u, v) = u^T C v$
 C positive definite.

→ Inner product in Non-Euclidean spaces

EX: $(f, g) \mapsto \int_a^b f(t)g(t) dt$ is a I.P. on
 $P_2(I) :=$ set of all poly of $\text{deg} \leq 2$ on $I = [a, b]$

(EXER) Check (\cdot, \cdot) is also I.P. on $C^k(I)$. $P(I)$.

Q What is the matrix representation of (\cdot, \cdot) on $P_2(I)$

A: $at^2 + bt + c \mapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\{t^2, t, 1\}$ standard basis
 $v_1 \quad v_2 \quad v_3$

3x3 sym matrices
 ↓
 6 entries

$$\begin{cases} (t^2, t^2) = \int_0^1 t^4 dt = \frac{1}{5} & (t, t) = \int_0^1 t^2 dt = \frac{1}{3} \\ (t^2, t) = \int_0^1 t^3 dt = \frac{1}{4} & (t, 1) = \int_0^1 t dt = \frac{1}{2} \\ (t^2, 1) = \int_0^1 t^2 dt = \frac{1}{3} & (1, 1) = \int_0^1 1 dt = 1 \end{cases}$$

$$f = a_2 t^2 + a_1 t + a_0$$

$$(f, g) = [a_2 \ a_1 \ a_0]$$

$$g = b_2 t^2 + b_1 t + b_0$$

$$\begin{bmatrix} \frac{1}{5} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix} \begin{matrix} \leftarrow v_1 \\ \leftarrow v_2 \\ \leftarrow v_3 \end{matrix} \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}$$

↑ ↑ ↑
 v_1 v_2 v_3

Def: A V.S. V with an IP is called an inner product space.
 if $\dim V < \infty$, V is called an Euclidean space.

length: $\|u\| := \sqrt{(u, u)}$

Distance: $d(u, v) = \|u - v\|$

Angle: $\cos \theta = \frac{(u, v)}{\|u\| \|v\|}$ $\cos \theta \in [-1, 1]$

Orthogonal: $(u, v) = 0$ Unit vector: $u = \frac{x}{\|x\|}$

Def: $S = \{v_1, \dots, v_n\} \subseteq V$ is called orthogonal if

① $(v_i, v_j) = 0 \quad i \neq j$

S is called orthonormal if in addition

② $\|v_i\| = 1$

EX1: $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is orthogonal.

$\tilde{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}$ $\tilde{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$ v_3 is orthonormal.

Thm: $S = \{v_1, \dots, v_n\}$ orthogonal $v_i \neq 0 \Rightarrow S \perp I$. (Give proof)

Cor S orthonormal $\Rightarrow S \perp I$. Remark: NOT True for orthogonal.

EX1 (Continuous) $\{v_1, v_2, v_3\}$ orthogonal basis $\{\tilde{v}_1, \tilde{v}_2, v_3\}$ orthonormal basis.

Properties: (1) $|(u, v)| \leq \|u\| \|v\|$ (Cauchy-Schwarz Inequality)

(2) $\|u + v\| \leq \|u\| + \|v\|$ (Triangle Inequality)

Example 14 Post