

EXAMPLES OF SECTIONS 5.6

Question 1. Find the eigenvalues and eigenvectors of the following matrices:

(a)

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}.$$

(b)

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Solutions.

a. Start with the characteristic polynomial

$$\det \begin{bmatrix} 3 - \lambda & -2 \\ 2 & -2 - \lambda \end{bmatrix} = -(3 - \lambda)(2 + \lambda) + 4 = 0,$$

whose solutions are the eigenvalues

$$\lambda_1 = 2, \lambda_2 = -1.$$

Let us find the corresponding eigenvectors.

$\lambda_1 = 2$:

$$\begin{bmatrix} 3 - \lambda_1 & -2 \\ 2 & -2 - \lambda_1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix},$$

hence we want to solve

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} v_1 = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find

$$v_1 = s \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

As we saw in class, we can drop the free variable s and write

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$\lambda_2 = -1$:

$$\begin{bmatrix} 3 - \lambda_2 & -2 \\ 2 & -2 - \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix},$$

hence we want to solve

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} v_2 = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find

$$v_2 = s \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Again, we drop the free variable s , obtaining

$$v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Summarizing, we have the following eigenvalues and eigenvectors:

$$\lambda_1 = 2, v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \lambda_2 = -1, v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

b. Start with the characteristic polynomial

$$\det \begin{bmatrix} 1 - \lambda & 1 & 2 \\ 1 & 2 - \lambda & 1 \\ 2 & 1 & 1 - \lambda \end{bmatrix} = (1 - \lambda)((2 - \lambda)(1 - \lambda) - 1) - (1 - \lambda - 2) + 2(1 - 2(2 - \lambda)) = 0.$$

Rearranging,

$$\begin{aligned} (2 - \lambda)(1 - \lambda)^2 - 1 + \lambda + 1 + \lambda - 6 + 4\lambda &= (2 - \lambda)(1 - \lambda)^2 - 6(1 - \lambda) \\ &= (1 - \lambda)((2 - \lambda)(1 - \lambda) - 6) = 0. \end{aligned}$$

The eigenvalues are now easily found to be

$$\lambda_1 = 4, \lambda_2 = -1, \lambda_3 = 1.$$

Let us find the corresponding eigenvectors.

$\lambda_1 = 4$:

$$\begin{bmatrix} 1 - \lambda_1 & 1 & 2 \\ 1 & 2 - \lambda_1 & 1 \\ 2 & 1 & 1 - \lambda_1 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix},$$

so we need to solve

$$\begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving the system and ignoring the free variable as before we obtain

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Repeating the process for $\lambda_2 = -1, \lambda_3 = 1$ we find, respectively

$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Summarizing, we have the following eigenvalues with corresponding eigenvectors

$$\lambda_1 = 4, v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \lambda_2 = -1, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \lambda_3 = 1, v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$