EXAMPLES OF SECTIONS 1.8

Question 1. Find the orthogonal trajectories of $x^2 + y^2 - 2Cx = 0$.

SOLUTIONS.

1. First, we differentiate the given family of curves with respect to x on both sides. This yields

$$2x + 2y\frac{dy}{dx} - 2C = 0 \Longrightarrow \frac{dy}{dx} = \frac{C - x}{y}. \tag{1}$$

By $x^2 + y^2 - 2Cx = 0$, we can solve C in terms of x, y and get $C = \frac{x^2+y^2}{2x}$. Plug this expression back into (1), we can get

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

Thus the slop of the orthogonal trajectories satisfy

$$\frac{dy}{dx} = -\frac{2xy}{y^2 - x^2} = \frac{2y/x}{1 - (y/x)^2}.$$

Making the change of variables $V = y/x$, the above DE can be transformed into

$$xV' + V = \frac{2V}{1 - V^2} \Longleftrightarrow xV' = \frac{2V}{1 - V^2} - V = \frac{V + V^3}{1 - V^2}.$$

This is separable DE. It can be written as

$$\int \frac{1 - V^2}{V(1 + V^2)} dV = \int \frac{dx}{x} + K = \ln|x| + K.$$

The LHS can be solved by partial fraction. More precisely, for two constants A, B to be determined later, the following holds

$$\frac{1 - V^2}{V(1 + V^2)} = \frac{A}{V} + \frac{B + DV}{1 + V^2} = \frac{A + AV^2 + BV + CV^2}{V(1 + V^2)}.$$

By comparing all the coefficients, it is not hard to conclude that $A = 1, B = 0, D = -2$. Therefore, the original problem reduces to

$$\begin{aligned}
&\int \frac{dV}{V} - \int \frac{2V}{1+V^2} dV = \int \frac{dx}{x} + K = \ln|x| + K \\
\Longleftrightarrow &\ln|V| - \ln(1+V^2) = \ln|x| + K \\
\Longleftrightarrow &\ln|\frac{V}{1+V^2}| = \ln|x| + K \\
\Longleftrightarrow &|\frac{V}{1+V^2}| = K|x| \\
\Longleftrightarrow &\frac{V}{1+V^2} = Kx \qquad (2) \\
\Longleftrightarrow &\frac{y/x}{1+(y/x)^2} = Kx \\
\Longleftrightarrow &\frac{xy}{x^2+y^2} = Kx \\
\Longleftrightarrow &\frac{y}{x^2+y^2} = K \\
\Longleftrightarrow &x^2 + y^2 = Kx.
\end{aligned}$$

Remark 0.1. *Consider why we can remove the absolutely values on the sides of the equation in* (2).