EXAMPLES OF SECTIONS 2.5

Definition 0.1. *The process of reducing* $[A|b]$ *to REF and then using back substitution to solve it is called* **Gaussian elimination**. *The process of reducing* $[A|b]$ *to RREF and solving it is called* **Gauss-Jordan elimination**.

Question 1. Use Gauss-Jordan elimination to solve the system:

$$\begin{cases} x + 3y + 2z = 2 \\ 2x + 7y + 7z = -1 \\ 2x + 5y + 2z = 7 \end{cases}$$

Question 2. Use Gauss-Jordan elimination to solve the system:

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 4x_4 + x_5 = 7 \\ 2x_1 + 6x_2 + 5x_4 + 2x_5 = 5 \\ 4x_1 + 11x_2 + 8x_3 + 5x_5 = 7 \\ x_1 + 3x_2 + 2x_3 + x_4 + x_5 = -2. \end{cases}$$

Question 3. Is

$$\begin{cases} x - 2y + 4z = a \\ -x + y - 3z = b \\ 4x + 3y + 5z = c \end{cases}$$

consistent for all values of a, b, c ?

SOLUTIONS.

1. The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 2 & 7 & 7 & -1 \\ 2 & 5 & 2 & 7 \end{array}\right]$$

Then

$$\begin{bmatrix} 1 & 3 & 2 & \vdots & 2 \\ 2 & 7 & 7 & \vdots & -1 \\ 2 & 5 & 2 & \vdots & 7 \end{bmatrix} \underset{\sim}{\overset{A_{12}(-2)}{A_{13}(-2)}} \begin{bmatrix} 1 & 3 & 2 & \vdots & 2 \\ 0 & 1 & 3 & \vdots & -5 \\ 0 & -1 & -2 & \vdots & 3 \end{bmatrix}$$

$$\underset{\sim}{A_{23}(1)} \begin{bmatrix} 1 & 3 & 2 & \vdots & 2 \\ 0 & 1 & 3 & \vdots & -5 \\ 0 & 0 & 1 & \vdots & -2 \end{bmatrix} \underset{\sim}{\overset{A_{32}(-3)}{A_{31}(-2)}} \begin{bmatrix} 1 & 3 & 0 & \vdots & 6 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & -2 \end{bmatrix}$$

$$\underset{\sim}{A_{21}(-3)} \begin{bmatrix} 1 & 0 & 0 & \vdots & 3 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & -2 \end{bmatrix}$$

Therefore the solution of the system is $x = 3$, $y = 1$, $z = -2$.

2.

$$\begin{bmatrix} 1 & 3 & -2 & 4 & 1 & \vdots & 7 \\ 2 & 6 & 0 & 5 & 2 & \vdots & 5 \\ 4 & 11 & 8 & 0 & 5 & \vdots & 3 \\ 1 & 3 & 2 & 1 & 1 & \vdots & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 4 & 1 & \vdots & 7 \\ 0 & 0 & 4 & -3 & 0 & \vdots & -9 \\ 0 & -1 & 16 & -16 & 1 & \vdots & -25 \\ 0 & 0 & 4 & -3 & 0 & \vdots & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 4 & 1 & \vdots & 7 \\ 0 & -1 & 16 & -16 & 1 & \vdots & -25 \\ 0 & 0 & 4 & -3 & 0 & \vdots & -9 \\ 0 & 0 & 4 & -3 & 0 & \vdots & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 4 & 1 & \vdots & 7 \\ 0 & -1 & 16 & -16 & 1 & \vdots & -25 \\ 0 & 0 & 4 & -3 & 0 & \vdots & -9 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 4 & 1 & \vdots & 7 \\ 0 & 1 & -16 & 16 & -1 & \vdots & 25 \\ 0 & 0 & 1 & -\frac{3}{4} & 0 & \vdots & -\frac{9}{4} \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -\frac{19}{2} & 4 & \vdots & \frac{71}{2} \\ 0 & 1 & 0 & 4 & -1 & \vdots & -11 \\ 0 & 0 & 1 & -\frac{3}{4} & 0 & \vdots & -\frac{9}{4} \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

So the solution can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{19}{2} \\ -4 \\ \frac{3}{4} \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} \frac{71}{2} \\ -11 \\ -\frac{9}{4} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Try to fill the elementary row operations in this Gauss-Jordan elimination!

3. The augmented matrix of the system is

$$\begin{bmatrix} 1 & -2 & 4 & \vdots & a \\ -1 & 1 & -3 & \vdots & b \\ 4 & 3 & 5 & \vdots & c \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & -2 & 4 & \vdots & a \\ -1 & 1 & -3 & \vdots & b \\ 4 & 3 & 5 & \vdots & c \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 & \vdots & a \\ 0 & 1 & -1 & \vdots & -a-b \\ 0 & 0 & 0 & \vdots & 7a+11b+c \end{bmatrix}$$

When $7a+11b+c=0$, $\text{rank}(A) = \text{rank}(A|b) = 2$. This system is consistent. But when $7a+11b+c \neq 0$, $2 = \text{rank}(A) < \text{rank}(A|b) = 3$. Therefore, this system is inconsistent in this case.