MORE EXAMPLES/COUNTEREXAMPLES OF VECTOR SPACES

Question 1. $(V, \oplus; \mathbb{R}, \odot)$ is the set of all $n \times n$ matrices with real entries equipped with $\oplus$ and $\odot$ defined in the following way:

$$A \oplus B = 2A + 2B, \quad c \odot A = cA^T.$$

Check whether this is a vector space.

Question 2. $(V, \oplus; \mathbb{R}, \odot)$ is the set of all positive real numbers with $\oplus$ and $\odot$ defined by

$$u \oplus v = uv, \quad c \odot u = u^c.$$

Check whether this is a vector space.

SOLUTIONS.

1. The set V itself coincide with $M_n(\mathbb{R})$. But the operations are defined in a different way.

(a) If $A, B \in V = M_n(\mathbb{R})$, then $A \oplus B = 2A + 2B \in V$. Holds.

(b) If $c \in \mathbb{R}$ and $A \in V$, then $c \odot A = cA^T \in V$. Holds.

Similarly, (1) holds true.

(2) $A \oplus (B \oplus C) = A \oplus (2B + 2C) = 2A + 4B + 4C$. But $(A \oplus B) \oplus C = (2A + 2B) \oplus C = 4A + 4B + 2C \neq A \oplus (B \oplus C)$. So (2) fails.

$(V, \oplus; \mathbb{R}, \odot)$ is not a vector space.

2. (a) If $u, v \in V$, then $u \oplus v = uv > 0$. So $u \oplus v \in V$. Holds.

(b) If $c \in \mathbb{R}$ and $u \in V$, then $c \odot u = u^c$. Holds.

(1) $u \oplus v = uv = vu = v \oplus u$. Holds.

(2) $u \oplus (v \oplus w) = u \oplus (vw) = uvw = (uv) \oplus w = (u \oplus v) \oplus w$. Holds.

(3) $1 \in V$ and $u \oplus 1 = u = 1 \oplus u$. Holds. $0_V = 1$.

(4) For any $u \in V$, $u \oplus \frac{1}{u} = 1 = 0_V$.

(5) $c \odot (u \oplus v) = c \odot (uv) = (uv)^c = u^c v^c = u^c \oplus v^c = (c \odot u) \oplus (c \odot v)$. Holds.

(6) $(c + d) \odot u = u^{c+d} = u^c u^d = u^c \oplus u^d = (c \odot u) \oplus (d \odot u)$. Holds.

(7) $(cd) \odot u = u^{cd} = (u^d)^c = (d \odot u)^c = c \odot (d \odot u)$. Holds.

(8) $1 \odot u = u^1 = u$. Holds.

So $(V, \oplus; \mathbb{R}, \odot)$ is a vector space.