EXAMPLES OF SECTIONS 5.2

Question 1. T is a linear transformation from $\mathbb{P}_1$ to $\mathbb{P}_2$. Moreover,
$$T(a + bx) = (2a - 3b) + (b - 5a)x + (a + b)x^2.$$
Find $\text{Ker}(T)$ and $\text{Rng}(T)$.

Solutions.

1. We identify T as a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. By the given conditions, we have
$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \quad T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}.$$
So the representation matrix $[T]$ of T is
$$\begin{bmatrix} -3 & 2 \\ 1 & -5 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
$\text{Ker}(T) = \text{Nullspace of} [T]$. For any $m \times n$ matrix A, $\text{rank}(A) + \dim(\text{Nullspace of } A) = n$. So $\text{Ker}(T) = \{0\}$.

On the other hand,
$$\begin{aligned}
\text{Rng}(T) &= \{T(ax + b) : a, b \in \mathbb{R}\} \\
&= \{(2a - 3b) + (b - 5a)x + (a + b)x^2 : a, b \in \mathbb{R}\} \\
&= \{a(2 - 5x + x^2) + b(-3 + x + x^2) : a, b \in \mathbb{R}\} \\
&= \text{span}\{2 - 5x + x^2, -3 + x + x^2\}.
\end{aligned}$$
$\dim(\text{Rng}(T)) = 2$.