EXAMPLES OF SECTION 6.5

Example 1. *A 16 lb object stretches a spring* 8/9 *ft by itself. There is no damping and no external forces acting on the system. The spring is initially displaced 6 inches upwards from its equilibrium position and given an initial velocity of 1 ft/sec downward. Find the displacement* $u(t)$ *at any time* t*.*

Solution. We first need to set up the equation for the problem. This requires us to find m and k.

The mass is

$$m = \frac{\text{Weight}}{g} = \frac{16}{32} = \frac{1}{2}.$$

Now we find k by Hooks law:

$$k = \frac{mg}{L} = \frac{16}{8/9} = 18.$$

We can now set up the IVP.

$$u'' + 36u = 0, \quad u(0) = -\frac{1}{2}, \quad u'(0) = 1.$$

For the initial conditions recall that upward displacement/motion is negative while downward displacement/motion is positive. Also, since we decided to do everything in feet we had to convert the initial displacement to feet.

The associated polynomial differential equation is

$$(D^2 + 36)u = 0,$$

which gives us the complex conjugate roots $r = \pm 6i$. The general solution can be found to be:

$$u = C_1 \cos(6t) + C_2 \sin(6t).$$

To find the undetermined constants, we plug in the initial values:

$$u(0) = C_1 = -\frac{1}{2},$$

and

$$u'(0) = -6C_1 \cos(0) + 6C_2 \sin(0) = 1 \Longrightarrow C_2 = \frac{1}{6}.$$

The displacement at any time t now is given by

$$u(t) = -\frac{1}{2}\cos(6t) + \frac{1}{6}\sin(6t).$$

Now, let's convert this to a single cosine. First let's get the amplitude, R.

$$R = \sqrt{(-\frac{1}{2})^2 + (\frac{1}{6})^2} = \frac{\sqrt{10}}{6}.$$

The phase is now given by

$$\phi = \cos^{-1}(\frac{-1/2}{\sqrt{10}/6}) = \cos^{-1}(-\frac{3}{\sqrt{10}}).$$

This angle is between $\pi/2$ to π, which makes $\sin\phi > 0$. So we are good, since we expect $\sin\phi = \frac{1/6}{\sqrt{10}/6} = \frac{1}{\sqrt{10}}$. This yields

$$u(t) = \frac{\sqrt{10}}{6}\cos(6t - \phi).$$

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Remark 2. *The difficulty of this kind of problems is to determine the phase angle. For example, if*

$$u(t) = -\frac{1}{2}\cos(6t) - \frac{1}{6}\sin(6t).$$

We still get the same amplitude $R = \frac{\sqrt{10}}{6}$*. So the angle*

$$\phi = \cos^{-1}(\frac{-1/2}{\sqrt{10}/6}) = \cos^{-1}(-\frac{3}{\sqrt{10}})$$

remain the same as in the example, and thus is between $\pi/2$ *to* π*. However, in this case our* $\sin\phi = \frac{-1/6}{\sqrt{10}/6} = -\frac{1}{\sqrt{10}} < 0$*. A contradiction. An angle that works will be*

$$\theta = 2\pi - \phi,$$

as cosine is an even function

$$\cos(\theta) = \cos(2\pi - \phi) = \cos(-\phi) = \cos(\phi) = -\frac{3}{\sqrt{10}}$$

and sine is an odd function

$$\sin(\theta) = \sin(2\pi - \phi) = \sin(-\phi) = -\sin(\phi) = -\frac{1}{\sqrt{10}}.$$