## EXAMPLES OF SECTIONS 7.2

**Question 1.** Transfor the given DE into an equivalent system of first order DEs:

$$t^3x''' - 2t^2x'' + 3tx' + 5x = \ln t.$$

**Question 2.** Write the system of question 1 in matrix form.

**Question 3.** Consider the system

$$\vec{x}' = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \vec{x}.$$

Determine whether the vectors

$$\vec{x}_1 = \begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix}$$

and

$$\vec{x}_2 = \begin{bmatrix} e^{2t} \\ -2e^{2t} \end{bmatrix}$$

are solutions of the system. What can you say about the general solution?

**SOLUTIONS.**

**1.** Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$, hence

$$x_3' = x''' = \frac{-5x - 3tx' + 2t^2x'' + \ln t}{t^3},$$

or

$$x_3' = -\frac{5}{t^3}x_1 - \frac{3}{t^2}x_2 + \frac{2}{t}x_3 + \frac{\ln t}{t^3}.$$

Therefore

$$\begin{cases} x_1' &= & & & x_2 & & & & \\ x_2' &= & & & & & x_3 & & \\ x_3' &= & -\frac{5}{t^3}x_1 & - & \frac{3}{t^2}x_2 & + & \frac{2}{t}x_3 & + & \frac{\ln t}{t^3} \end{cases}$$

**2.** We readily see that if

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and

$$\vec{b} = \begin{bmatrix} 0 \\ 0 \\ \frac{\ln t}{t^3} \end{bmatrix}$$

then

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{5}{t^3} & -\frac{3}{t^2} & \frac{2}{t} \end{bmatrix} \vec{x} + \vec{b}.$$

**3.** First compute

$$\vec{x}_1' = \begin{bmatrix} (e^{3t})' \\ (-e^{3t})' \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ -3e^{3t} \end{bmatrix}.$$

But

$$\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \vec{x}_1 = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix} = \begin{bmatrix} 4e^{3t} - e^{3t} \\ -2e^{3t} - e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ -3e^{3t} \end{bmatrix}$$

so that

$$\vec{x}_1' = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \vec{x}_1$$

and $\vec{x}_1$ is a solution.

Let us verify that $\vec{x}_2$ is a solution in a different way. Write

$$\vec{x}_2 = \begin{bmatrix} e^{2t} \\ -2e^{2t} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{2t}.$$

Taking the derivative we have

$$\vec{x}_2' = \begin{bmatrix} 1 \\ -2 \end{bmatrix} (e^{2t})' = \begin{bmatrix} 1 \\ -2 \end{bmatrix} 2e^{2t} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} e^{2t}.$$

Now compute

$$\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \vec{x}_2 = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{2t} = \begin{bmatrix} 4-2 \\ -2-2 \end{bmatrix} e^{2t} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} e^{2t},$$

so that

$$\vec{x}_2' = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \vec{x}_2$$

and $\vec{x}_2$ is a solution.

Let us verify now that $x_1$ and $x_2$ are linearly independent. Their Wronskian is

$$W(t) = \det \begin{bmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{bmatrix} = e^{3t}(-2e^{2t}) - e^{2t}(-e^{3t}) = -e^{5t} \neq 0,$$

hence the solutions are linearly independent. Since the system is $2 \times 2$, it admits at most two linearly independent solutions. We conclude that the general solution is

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2.$$