MA 26200 FINAL EXAM (GREEN) FALL 2015

Name ______________________________ **PUID#** ____________________

Section#____________ **Class Time** ____________ **Lecturer** ______________________

Exam Rules

1. You may not open the exam until instructed to do so.
2. You must obey the orders and requests by all proctors, TAs, and lecturers.
3. You may not leave during the first 20 min or during the last 10 min of the exam.
4. No books, notes, calculators, or any electronic devices are allowed on the exam, and they should not even be in sight in the exam room. Phones are to be turned off. You may not look at anybody else's test, and may not communicate with anybody else except, if you have a question, with a TA or lecturer.
5. After time is called, you must put down all writing instruments and remain in your seat, while the TAs collect the scantrons and the exams.
6. Any violation of these rules and any act of academic dishonesty may result in severe penalties. All violators will be reported to the Office of the Dean of Students.

I have read and understood the exam rules stated above:

STUDENT SIGNATURE: ____

Instructions

1. When told to begin, make sure you have a complete test. There are **14** different test pages, including this cover page. There are 25 problems. Each problem is worth 8 points. The maximum possible score is 200 points. Make sure that you have a **green** answer sheet. Fill in the information requested above. Your PUID# is your student identification number.
2. **Using a #2 pencil**, fill in each of the following items on your **answer sheet**:
   (a) On the top left side, print your name (last name, first name), and fill in the little circles.
   (b) On the bottom left side, under SECTION NUMBER, write in your **4** digit section number (for example 0012 or 0003) and fill in the little circles. **The section numbers are listed below.**
   (c) On the bottom, under STUDENT IDENTIFICATION NUMBER, write in your student I.D. number and fill in the little circles.
   (d) On the bottom right, print your **instructor's name** and the **course number**.
   (e) SIGN your answer sheet. **Write 01** in the TEST/QUIZ NUMBER boxes and **fill in** the appropriate little circles.
3. Do any necessary work for each problem in the space provided or on the back of the pages of this test. No partial credit is given but your work may be considered if your grade is borderline. Circle your answers on this test.
4. **Using a #2 pencil**, put your answers to questions 1–25 on your answer sheet by filling in the circle of the letter of your response. Double check that you have filled in the circles you intended. If more than one circle is filled in for any question, your response will be considered incorrect.
7. Hand in your answer sheet **and** this test to your lecturer or TA.

Here is a list of the section numbers:

011 - UNIV 119 MWF 02:30pm - **Banerjee,** Arindam
041 - UNIV 101 TR 04:30am - **Hedayatzadeh,** M. Hadi
071 - UNIV 219 MWF 03:30pm - **Banerjee,** Arindam
111 - UNIV 117 MWF 11:30pm - **Zhang,** Ying
133 - PHY 338 MWF 08:30am - **Shao,** Yuanzhen
031 - MSEE B010 MWF 09:30am - **Yeung,** Sai Kee
061 - UNIV 101 TR 03:00pm - **Hedayatzadeh,** M. Hadi
081 - UNIV 117 MWF 12:30pm - **Zhang,** Ying
131 - PHY 338 MWF 09:30am - **Shao,** Yuanzhen

1. Let y be a solution to the initial value problem $\frac{dy}{dx} + \frac{y}{x} = e^x$, $\quad y(1) = 2$. What is the value of $y(2)$?

A. $e^2 + 2$

B. $\frac{e}{2}$

C. $\frac{e^2+2}{2}$

D. $\frac{e-1}{2}$

E. $e \ln 2 + 2$

This is a first order Linear DE.

$I(x) = e^{\int \frac{1}{x} dx} = x$

$\int (xy)' = \int x e^x dx = xe^x - e^x + C$

$y = e^x - \frac{e^x}{x} + \frac{C}{x}$

$y(1) = e - e + C = 2 \Rightarrow C = 2$

$y(2) = e^2 - \frac{e^2}{2} + 1$

2. Solutions to $\underbrace{(2xy + \cos y)}_{M}dx + \underbrace{(x^2 - x\sin y - 2)}_{N}dy = 0$ satisfy

A. $x^2y + x\cos y - 2y = c$

B. $x^2y + \cos y - 2y = c$

C. $x^2y + x\sin y - 2 = c$

D. $x^2y - x\cos y - 2x = c$

E. $x^2y^2 + x\cos y - 2x = c$

$\frac{\partial}{\partial y} M = 2x - \sin y = \frac{\partial}{\partial x} N$ Exact

$\Phi = \int M\, dx = x^2 y + x\cos y + h(y)$

$\frac{\partial}{\partial y}\Phi = x^2 - x\sin y + h'(y) = x^2 - x\sin y - 2$

$\Downarrow$

$h'(y) = -2 \Rightarrow h(y) = -2y$

3. The general solution to $y'' = 2x^{-1}y' + 4x^2$ is

A. $y(x) = c_1x^3 + c_2x^4$
B. $y(x) = x^7 + c_1x^3 + c_2$
C. $y(x) = x^3 + c_1x^4$
D. $y(x) = c_1x^3 + x^4 + c_2$
E. $y(x) = x^3 + c_1$

Let $u = y'$

$u' - \frac{2}{x} u = 4x^2$ 1st order linear

$I(x) = e^{\int -\frac{2}{x} dx} = \frac{1}{x^2}$

$\left(\frac{1}{x^2} u\right)' = 4 \Rightarrow u = 4x^3 + C_1x^2$

$y' = 4x^3 + C_1x^2$

$y = x^4 + C_1x^3 + C_2$

4. General solution to $(1+y^2)\frac{dy}{dx} = \cos x$ is

A. $y^3 + 3y = \sin x + c$
B. $y^3 + 3y = 3\sin x + c$
C. $y^3 + y = \sin x + c$
D. $y^3 + y = 3\sin x + c$
E. None of the above.

Separable equation

$\int (1+y^2)\, dy = \int \cos x \, dx$

$y + \frac{1}{3}y^3 = \sin x + C.$

$y^3 + 3y = \sin x + C$

5. Only one of the following is NOT always true. Which one is it?

A. The product of two 3×3 diagonal matrices is a diagonal matrix.
B. For any two $n \times n$ matrices A, B, $(A+B)^2 = A^2 + BA + AB + B^2$.
C. For any two matrices A, B, if $AB = 0$ then either $A = 0$ or $B = 0$.
D. Product of two 3×3 upper triangular matrices is an upper triangular matrix.
E. Transpose of a symmetric matrix is symmetric.

C. $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a counterexample.

6. A tank initially contains 20L of water. A solution containing 5 g/L of salt flows into the tank at a rate of 11 ~~8~~L/min., and the well stirred mixture flows out at a rate of 3 L/min. Which of the following describes $A(t)$, the amount of salt in the tank at time t before the tank becomes full?

A. $\frac{dA}{dt} = \cancel{15}\,55 - \frac{A}{20+8t}$, $A(0) = 10$
B. $\frac{dA}{dt} = \cancel{40}\,55 - \frac{A}{20+3t}$, $A(0) = 0$
C. $\frac{dA}{dt} = \cancel{15}\,55 - \frac{A}{8+8t}$, $A(0) = 0$
D. $\frac{dA}{dt} = \cancel{40}\,55 - \frac{3A}{20+8t}$, $A(0) = 0$
E. $\frac{dA}{dt} = \cancel{40}\,55 - \frac{3A}{20+3t}$, $A(0) = 10$

$c_1 = 5 \quad r_1 = 11$

$r_2 = 3$

$V(0) = 20 \quad A(0) = 0$

$V = 20 + 8t$

$$A' = c_1 r_1 - c_2 r_2$$

$$= 55 - \frac{3A}{V} = 55 - \frac{3A}{20+8t}$$

7. For what values of k the following system of linear equations has **infinitely many** solutions?

$$\begin{array}{rcrcrcl} 4x_1 & - & 3x_2 & & & = & k-3 \\ (k-1)x_1 & & & + & 3x_3 & = & -3 \\ (k-1)x_1 & + & 3x_2 & + & (k+3)x_3 & = & -8 \end{array}$$

A. $k=-3$
B. $k=4$
C. $k=4,-3$
D. $k\neq -3$
E. $k\neq 4,-3$

$$0 = \det\begin{bmatrix} 4 & -3 & 0 \\ k-1 & 0 & 3 \\ k-1 & 3 & k+3 \end{bmatrix} = \begin{vmatrix} 4 & -3 & 0 \\ k-1 & 0 & 3 \\ 0 & 3 & k \end{vmatrix} = \begin{vmatrix} 4 & 0 & k \\ k-1 & 0 & 3 \\ 0 & 3 & k \end{vmatrix}$$

$$= 3(-1)\begin{vmatrix} 4 & k \\ k-1 & 3 \end{vmatrix} = 3(k^2-k-12)$$

$$= 3(k-4)(k+3)$$

When $k=4$,

the system has ∞ many sol.

When $k=-3$,

the system has no solution.

8. Find k such that $\begin{bmatrix} 5 & 2 \\ -2 & k-1 \end{bmatrix}$ is in the span$\{\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\}$

A. 2
B. 4
C. 7
D. 8
E. 11

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 5 \\ 1 & 1 & 3 & 2 \\ 2 & 2 & 0 & -2 \\ 2 & 0 & 2 & k-1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -1 & 2 & 5 \\ 0 & 0 & -6 & -6 \\ 2 & 0 & 2 & k-1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & k-3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & k-7 \end{array}\right]$$

$k-7=0$

9. If $A = \begin{bmatrix} 0 & 2 & -2 \\ 2 & 3 & 0 \\ 1 & 5 & -3 \end{bmatrix}$. What is the **sum** of the entries in the third row of A^{-1}?

A. $-\frac{5}{2}$
B. $\frac{5}{2}$
C. $\frac{3}{2}$
D. 1
E. 5

$$\frac{C_{13} + C_{23} + C_{33}}{|A|} = \frac{7 + 2 - 4}{\begin{vmatrix} 0 & 2 & -2 \\ 0 & -7 & 6 \\ 1 & 5 & -3 \end{vmatrix}} = \frac{5}{-2}$$

10. A is an $m \times n$ matrix and $\mathbf{b}$ is an $m \times 1$ vector. The equation $A\mathbf{x} = \mathbf{b}$ has **infinitely many** solutions. Consider the following statements:

(i) $m \leq n$
(ii) $n \leq m$
(iii) the rank of $A = n$
(iv) the rank of $A < n$
(v) $\det A = 0$

Which **must** be true?

A. only (i) and (v)
B. only (iv)
C. only (v)
D. only (iii) and (v)
E. None of the statements has to be true

Counterexamples: (i) $\begin{cases} x + y = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$ (ii) $x + y = 0$

(iii) $x + y = 0$

(v) A is not necessarily square.

11. Which of the following sets S are subspaces of the given vector space V?

(i) $S = \{A \in V \,:\, \mathrm{tr}(A) = 0\}$, $V = M_3(\mathbb{R}) = \{3 \times 3 \text{ matrices with real entries}\}$

(ii) $S = \{f \in V \,:\, f'' + x^2 f' - xe^x(f+1) = 0\}$, $V = C^2(\mathbb{R})$. → Solution to a nonhomogeneous DE.

(iii) $S = \{A \in V \,:\, A^2 + A = 0\}$, $V = M_2(\mathbb{R}) = \{2 \times 2 \text{ matrices with complex entries}\}$

(iv) $S = \{(x, y, z) \in V \,:\, 3x - y = 7z\}$, $V = \mathbb{R}^3$

(v) $S = \{$ solutions to the equation $\begin{bmatrix} 1 & 2 & -5 \\ 11 & -1 & 0 \\ 9 & -5 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 12 \\ 14 \end{bmatrix} \}$, $V = \mathbb{R}^3$. → inhomogeneous system Not a subspace.

A. only (iv)

B. only (i) and (iv)

C. only (i), (iii) and (v)

D. only (iii), (iv) and (v)

E. All of the above

(iii) A counterexample is

$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in S$

But $(2A)^2 + (2A) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \neq 0$

So $2A \notin S$.

12. Which of the following set of vectors forms a basis for $\mathbb{R}^3$?

A. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ → $\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$ a zero row

B. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ → $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$ two identical rows

D. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \right\}$ → contains a zero vector

E. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}$ → # of vectors > 3

13. Determine all values of k so that the vectors $(2, -k, 1), (1, -1, 1), (0, 1, -k)$ are **linearly dependent**.

A. $k \neq 1$
B. $k \neq 2$
C. $k \neq -1$
D. $k = 1$
E. $k = -1$

$$0 = \begin{vmatrix} 2 & 1 & 0 \\ -k & -1 & 1 \\ 1 & 1 & -k \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 2-k & 0 & 1 \\ -1 & 0 & -k \end{vmatrix}$$

$$= -\begin{vmatrix} 2-k & 1 \\ -1 & -k \end{vmatrix} = -(k^2 - 2k + 1)$$

$$\Downarrow$$

$$k = 1$$

14. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation for which

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \text{then } T\left(\begin{bmatrix} 1 \\ -5 \end{bmatrix}\right) =$$

A. $\begin{bmatrix} -8 \\ -5 \end{bmatrix}$
B. $\begin{bmatrix} 8 \\ 5 \end{bmatrix}$
C. $\begin{bmatrix} -8 \\ 5 \end{bmatrix}$
D. $\begin{bmatrix} 8 \\ -5 \end{bmatrix}$
E. $\begin{bmatrix} -5 \\ -8 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ -5 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$T\begin{pmatrix} 1 \\ -5 \end{pmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

15. The sum of the eigenvalues of the matrix $\begin{bmatrix} 7 & 4 \\ -1 & 8 \end{bmatrix}$ is

A. 1
B. 5
C. 10
D. 15
E. 20

$tr A = 7 + 8 = 15$

16. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ -1 & -2 & -3 & -1 \end{bmatrix},$$

the dimension of $nullspace(A)$ is :

A. 4
B. 3
C. 2
D. 1
E. 0

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank $A = 1$

nullity of $A = 4 -$ rank $A = 3$

17. Determine the general solution to

$$(D^2 - 1)(D^2 + 6D + 13)^2 y = 0.$$

A. $c_1 e^x + e^{-3x}[c_2 \cos(2x) + c_3 \sin(2x) + c_4 x \cos(2x) + c_5 x \sin(2x)]$
B. $c_1 \cos x + c_2 \sin x + e^{-3x}[c_3 \cos(2x) + c_4 \sin(2x) + c_5 x \cos(2x) + c_6 x \sin(2x)]$
C. $c_1 e^x + c_2 e^{-x} + e^{-3x}[c_3 \cos(2x) + c_4 \sin(2x) + c_5 x \cos(2x) + c_6 x \sin(2x)]$
D. $c_1 e^x + e^{-3x}[c_2 \cos(2x) + c_3 \sin(2x)]$
E. $c_1 e^x + c_2 e^{-x} + e^{-3x}[c_3 \cos(2x) + c_4 \sin(2x)]$

$(D-1)(D+1)\ (D+3-2i)(D+3+2i)$

$r_1 = 1$ $\quad e^x$

$r_2 = -1$ $\quad e^{-x}$

$z_{1,2} = -3 \pm 2i$ $\quad e^{-3x}\cos 2x \quad x e^{-3x} \cos 2x$
$m = 2$ $\quad e^{-3x} \sin 2x \quad x e^{-3x} \sin 2x$

18. The general solution to

$$y''' - y'' + 2y = 0$$

is

A. $c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$
B. $c_1 e^{-x} + c_2 e^x \cos x + c_3 e^x \sin x$
C. $c_1 e^x + c_2 \cos x + c_3 \sin x$
D. $c_1 e^x + c_2 e^{-x} + c_3 x e^{-x}$
E. $c_1 e^x + c_2 e^{-x} \cos x + c_3 e^{-x} \sin x$

$(D^3 - D^2 + 2) = (D+1)(D^2 - 2D + 2)$

$r = 1$ $\quad e^x$

$z_{1,2} = 1 \pm i$ $\quad e^x \cos x$
$e^x \sin x$

19. To find a particular solution of the inhomogeneous differential equation

$$(D-3)(D^2+4)(D-1)^2 y = \underbrace{3xe^x}_{F_1} + \underbrace{\cos(2x)}_{F_2}$$

one can use the following trial solution

A. $(C_1 + C_2 x)e^x + C_3\cos(2x) + C_4\sin(2x)$
B. $(C_1 x + C_2 x^2)e^x + C_3 x\cos(2x) + C_4 x\sin(2x)$
C. $(C_1 x^2 + C_2 x^3)e^x + C_3 x\cos(2x) + C_4 x\sin(2x) + C_5 e^{3x}$
D. $C_1 e^{-x} + C_2\cos(3x) + C_3\sin(3x)$
E. $(C_1 x^2 + C_2 x^3)e^x + C_3 x\cos(2x) + C_4 x\sin(2x)$

$A_1(D) = (D-1)^2$ $y_{T_1} = x^2(A+Bx)e^x$

$A_2(D) = (D^2+4)$ $y_{T_2} = x(C\cos 2x + D\sin 2x)$

20. Using the variation-of-parameters method, we know that a particular solution to the differential equation

$$y'' + 4y = 4\csc(2x)$$

is $y_p(x) = u_1(x)\cos(2x) + u_2(x)\sin(2x)$. Then $u_2(x) =$

A. $\ln|\sin(2x)|$
B. $\sin(2x)$
C. 1
D. $\ln|\cos(2x)|$
E. $\sec(2x)$

$$W(y_1, y_2) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$u_2' = \frac{\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & 4\csc 2x \end{vmatrix}}{2} = 2\cot 2x$$

$$u_2 = 2\int \cot 2x\,dx = \ln|\sin 2x|$$

21. The motion of a spring-mass system is governed by the initial value problem

$$10x'' + 140x = 0, \quad x'(0) = 1, \quad x(0) = 0.$$

What is the amplitude of this motion?

A. $\frac{2\pi}{\sqrt{7}}$

B. $\frac{\sqrt{15}}{\sqrt{14}}$

C. $\frac{\sqrt{14}}{14}$

D. 1

E. $\sqrt{5}$

$X'' + 14X = 0 \qquad (D^2+14)X = 0$

$X = C_1 \cos(\sqrt{14}x) + C_2 \sin(\sqrt{14}x)$

$X(0) = C_1 = 0$

$X'(0) = \sqrt{14}\, C_2 \cos(\sqrt{14}x) = 1 \quad \Rightarrow \quad C_2 = \frac{\sqrt{14}}{14}$

$X = \frac{\sqrt{14}}{14} \sin(\sqrt{14}\,x)$

Amplitude $= \frac{\sqrt{14}}{14}$

22. One of solution to the differential equation

$$2t^2y'' + ty' - 3y = 0$$

is $y_1(t) = t^{-1}$. Another solution is of the form $y_2(t) = v(t)y_1(t)$ where v satisfies the differential equation

A. $2tv'' - v' = 0$

B. $t^2v'' + v' = 0$

C. $t^2v'' + 2v' = 0$

D. $(2t+1)v'' - 4v' = 0$

E. $2tv'' - 3v' = 0$

$2t^2y_2'' = 2t^2\left(v''\frac{1}{t} + 2v'\left(-\frac{1}{t^2}\right) + \cancel{v\left(\frac{2}{t^3}\right)}\right)$

$+\ ty' = t\left(v'\frac{1}{t} + \cancel{v\left(-\frac{1}{t^2}\right)}\right)$

$-\ 3y = \cancel{-3v\frac{1}{t}}$

$\|$

0

$2tv'' - 4v' + v' = 0$

23. The solution $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ to $\mathbf{x}'(t) = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} \mathbf{x}(t)$ satisfying $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has $x_1(\pi) =$

A. 0
B. 1
C. $\sqrt{2}$
D. -1
E. $-\sqrt{2}$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$$

$$A - iI = \begin{bmatrix} 0 & -1 \\ 1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \quad V_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad X(t) = C_1 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + C_2 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$X(0) = \begin{bmatrix} C_2 \\ C_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad X_1(\pi) = 1 \cdot (-\sin(\pi)) + 1 \cdot \cos(\pi) = -1$$

24. The real 2×2 matrix A has an eigenvalue $\lambda_1 = -\frac{1}{2} + i$ with corresponding eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Then the general solution of the system of differential equations

$$\mathbf{x}'(t) = A\,\mathbf{x}(t)$$

is $\mathbf{x}(t) =$

A. $C_1\, e^{t/2} \begin{bmatrix} -2\cos t \\ \sin t \end{bmatrix} + C_2\, e^{t/2} \begin{bmatrix} \sin t \\ -2\cos t \end{bmatrix}$

B. $C_1\, e^{-t/2} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + C_2\, e^{-t/2} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

C. $C_1\, e^{t} \begin{bmatrix} \cos t \\ -2\sin t \end{bmatrix} + C_2\, e^{t} \begin{bmatrix} \sin t \\ 2\cos t \end{bmatrix}$

D. $C_1\, e^{t/2} \begin{bmatrix} \cos t \\ 2\sin t \end{bmatrix} + C_2\, e^{t/2} \begin{bmatrix} -\sin t \\ 2\cos t \end{bmatrix}$

E. $C_1\, e^{t} \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} + C_2\, e^{t} \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix}$

$$\lambda_1 = -\frac{1}{2} + i$$

$$a = -\frac{1}{2} \quad b = 1$$

$$V_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{W} + i \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{U}$$

$$X = C_1 e^{at} [\cos bt\, W - \sin bt\, U]$$
$$+ C_2 e^{at} [\cos bt\, U + \sin bt\, W]$$

25. The system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ 3e^{-t} \end{bmatrix}$$

has fundamental matrix

$$\Psi(t) = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}.$$

A particular solution is $\mathbf{x}_p(t) = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ where $u_2 =$

A. 2

B. $-e^{t}$

C. $\frac{5t}{2}$

D. e^{-t}

E. $-\frac{1}{2}e^{-2t}$

$$u_2' = \frac{\begin{vmatrix} e^{-3t} & 2e^{-t} \\ -e^{-3t} & 3e^{-t} \end{vmatrix}}{\begin{vmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{vmatrix}} = \frac{5e^{-4t}}{2e^{-4t}} = \frac{5}{2}$$

$$u_2 = \frac{5}{2}t$$