NAME: ______________________________

INSTRUCTOR'S NAME: ______________________________

1. There are a total of 25 problems. You should show work on the exam sheet, and pencil in the correct answer on the scantron.
2. No books, notes, or calculators are allowed.

1. For what value of h is the following linear system CONSISTENT?

$$x - 3y = h$$
$$-2x + 6y = -5$$

A. $h = 0$
B. $h = 5/2$
C. $h = -5/2$
D. $h = 5$
E. $h = -5$

$$\left[\begin{array}{cc|c} 1 & -3 & h \\ -2 & 6 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 0 & 2h-5 \end{array}\right] \Rightarrow 2h - 5 = 0$$

2. For what values of r and s is the linear system INCONSISTENT?

$$x + y + z = 1$$
$$x \quad + 3z = -2 + s$$
$$x - y + rz = 3$$

A. $r = 5$ and $s = 4$
B. $r \neq 5$ and $s = 4$
C. $r = 5$ and $s \neq 4$
D. $r \neq 5$ and $s \neq 4$
E. None of the above

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & -2+s \\ 1 & -1 & r & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & -3+s \\ 0 & -2 & r-1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & -3+s \\ 0 & 0 & r-5 & 8-2s \end{array}\right]$$

when $r = 5$
$8 - 2s \neq 0$ this system is Inconsistent

3. Suppose A and B are $n \times n$ matrices. Which of the following statements are always TRUE?

i) $(-A)^T = -(A^T)$

ii $(A+B)^T = A^T + B^T$

iii) $(A^T B)^T = AB^T$

A. i) only

B. ii) only

C. i) and ii) only

D. ii) and iii) only

E. i), ii), and iii)

(iii) $(A^T B)^T = B^T A$

In general, B^T & A don't commute.

Counterexample : $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

4. Which of the following statements are always TRUE?

i) If a linear system $A\mathbf{x} = \mathbf{b}$ has m equations and n unknowns, and $m < n$, then the system must have infinitely many solutions.

ii) If A and B are $n \times n$ matrices and AB is nonsingular, then both A and B must be nonsingular.

iii) If A, B, and C are $n \times n$ matrices such that $AB = AC$, then $B = C$.

A. i) only

B. ii) only

C. i) and ii) only

D. ii) and iii) only

E. i), ii), and iii)

(i) $AX = b$ might have no solution

$$x + y + z = 1$$
$$0 = 1.$$

(ii) $|AB| \neq 0$

$$\begin{array}{c} |AB| \\ \| \\ |A||B| \end{array} \Rightarrow |A| \neq 0 \ \& \ |B| \neq 0$$

(iii) Counterexample

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

5. Which of the following are always true for real square matrices A?

i) if λ is an eigenvalue for A then $-\lambda$ is an eigenvalue for $-A$.

ii) if $\mathbf{v}$ is an eigenvector for A then $\mathbf{v}$ is also an eigenvector for $2A$.

iii) The eigenspace for the matrix A for the eigenvalue λ has dimension equal to the multiplicity of the root λ in the characteristic polynomial $p(\lambda) = \det(\lambda I - A)$.

A. i) and iii) only

B. i) only

C. iii) only

D. i), ii) and iii) only

E. i) and ii) only

(iii) Counterexample

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

6. Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v_4} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Which of the following statements are true?

i) The set $\{\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3},\ \mathbf{v_4}\}$ is linearly independent.

ii) The set $\{\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3},\ \mathbf{v_4}\}$ spans R^3.

iii) The set $\{\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3},\ \mathbf{v_4}\}$ is a basis for R^3.

Too many vectors

A. All of the above

B. None of the above

C. i) only

D. ii) only

E. ii) and iii) only

$$\begin{bmatrix} 1 & 4 & 5 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & 1 \\ 0 & -4 & -4 & 0 \\ 0 & -4 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 5 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ has rank 3.

7. For the inverse of the matrix $\begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, the entry in the second row and first column is

use adj A

A. 0

B. 1

C. -1

D. 1/2

E. -1/2

$$A_{12} = -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$$

$$|A| = 2 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -2$$

$$(A^{-1})_{21} = \frac{A_{12}}{|A|} = \frac{1}{2}$$

8. Given that

$$\det \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}}_{A} = 5,$$

what is the determinant of

$$\underbrace{\begin{bmatrix} a_4 & a_2 & a_3 & a_1 \\ b_4 & b_2 & b_3 & b_1 \\ 2c_4 + 3a_4 & 2c_2 + 3a_2 & 2c_3 + 3a_3 & 2c_1 + 3a_1 \\ d_4 & d_2 & d_3 & d_1 \end{bmatrix}}_{B} ?$$

A. 30

B. -30

C. 10

D. -10

E. 5

$$|A| = \frac{1}{2} \begin{vmatrix} a_1 & \cdots & a_4 \\ b_1 & \cdots & b_4 \\ 2c_1 & \cdots & 2c_4 \\ d_1 & \cdots & c_4 \end{vmatrix}$$

$$= -\frac{1}{2} \begin{vmatrix} a_4 & a_2 & a_3 & a_1 \\ b_4 & b_2 & b_3 & b_1 \\ 2c_4 & 2c_2 & 2c_3 & 2c_1 \\ d_4 & d_2 & d_3 & d_1 \end{vmatrix} = -\frac{1}{2} |B|$$

9. If A is a 3×3 matrix with $\det A = 3$, and $B = 2A$, what is $\det(A^T B^{-1})$?

A. 18

B. 1/18

C. 1/8

D. 1/2

E. 24

$$|A^T B^{-1}| = |A^T|\,|B^{-1}| = |A| \frac{1}{|B|} = |A| \frac{1}{|2A|}$$

$$= |A| \frac{1}{8|A|} = \frac{1}{8}$$

10. Compute the value of the following determinant:

$$\underbrace{\det \begin{bmatrix} 0 & 3 & 2 & 0 & 0 \\ 3 & 2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 4 & 2 & 0 & 2 & 4 \\ 0 & 3 & 2 & 0 & 5 \end{bmatrix}}_{A}$$

A. -30

B. -15

C. 0

D. 15

E. 30

$$|A| = 1 \cdot \begin{vmatrix} 0 & 3 & 0 & 0 \\ 3 & 2 & 2 & 3 \\ 4 & 2 & 2 & 4 \\ 0 & 3 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 0 & 0 \\ 3 & 2 & 2 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

$$= -3 \cdot \begin{vmatrix} 3 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= -15 \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} = 30$$

11. The dimension of the subspace of R_5 which is spanned by $[1,2,3,4,5]$, $[0,2,0,0,0]$, $[0,0,0,0,3]$, $[2,4,6,8,13]$, and $[2,6,6,8,10]$ is

A. 1

B. 2

C. 3

D. 4

E. 5

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 2 & 4 & 6 & 8 & 13 \\ 2 & 6 & 6 & 8 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank $A = 3$.

12. Let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by:

$$L([x,y,z]) = [ax^2 + bx, cy + z, d]$$

Which of the following choices of the parameters a, b, c, d gives a linear transformation?

A. $a = 1,\ b = 2,\ c = 3,\ d = 0$

B. $a = 0,\ b = 1,\ c = 0,\ d = 1$

C. $a = 0,\ b = 2,\ c = 4,\ d = 0$

D. $a = 1,\ b = 0,\ c = 0,\ d = 1$

E. $a = b = c = d = 1$

To make L a L.T. we need

$a = 0$

$d = 0$.

13. Let W be the subspace of $\mathbf{R}^3$ spanned by these two are colinear

$$\left\{ \underset{u_1}{\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

If we apply the Gram-Schmidt procedure to this set to obtain an orthonormal basis for W, we obain the set

We only need to use u_1 & u_2 to construct an orthonormal basis.

A. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

B. $\left\{ \frac{1}{8}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

C. $\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

D. $\left\{ \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

E. $\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$V_1 = U_1$

$V_2 = U_2 - \frac{(U_2, V_1)}{\|V_1\|^2} V_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

normalize V_1 & V_2. we obtain

$W_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$W_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

14. Let A be an $n \times n$ nonsingular matrix. Which of the following statements must be true?

i) $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

ii) A must be row equivalent to the identity matrix.

iii) A has rank n.

iv) $\det A \neq 0$.

(i) $AX = 0$ has only the trivial solution.

A. All are all true.

B. i), ii), and iii) only

C. iii) and iv) only

D. ii), iii), and iv) only

E. i), iii), and iv) only

15. Let A be an $m \times n$ matrix. Which of the following statements must be true regarding the nullity (dimension of the null space) of A?

i) The nullity of A is the same as the number of nonzero rows in a row echelon form of A.

ii) The nullity of A is the same as the dimension of the column space of A.

iii) The nullity of A is the same as the nullity of A^T.

A. None of the statements are correct.

B. i) and ii) only

C. i) and iii) only

D. ii) and iii) only

E. All three statements are correct.

Counterexamples:

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

16. Let A and B be $n \times n$ matrices. Which of the following statements must be true?

i) If B can be obtained from A by a sequence of row operations, and λ is an eigenvalue for A, then λ is also an eigenvalue for B.

ii) $\det(A + B) = \det A + \det B$.

iii) If A and B are both nonsingular, then $A + B$ is nonsingular.

A. None of the statements are correct.

B. All of the statements are correct.

C. i) only

D. i) and ii) only

E. i) and iii) only

Counterexamples

(i) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = -A$

17. Which of the following subsets of R_3 are subspaces?

i) The set of all vectors $[x, y, z]$ satisfying the condition $x + 2y - 3z = 1$

ii) The set of all vectors $[x, y, z]$ satisfying the condition $z = xy$

iii) The set of all vectors $[x, y, z]$ satisfying the condition $x = y$

A. i) only

B. ii) only

C. iii) only

D. i) and ii) only

E. ii) and iii) only

Counterexamples

(i) $(1, 0, 0)$ satisfies $x+2y-3z=1$

But $2(1, 0, 0)$ NOT.

(ii) $(1, 1, 1)$ satisfies $z = xy$

But $2(1, 1, 1)$ NOT.

18. Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Which of the following are true?

i) $\lambda = 2$ is an eigenvalue whose eigenspace has dimension 1.

ii) $\lambda = 1$ is an eigenvalue whose eigenspace has dimension 2.

iii) $\lambda = 0$ is an eigenvalue whose eigenspace has dimension 2.

A. i) only

B. ii) only

C. iii) only

D. ii) and iii) only

E. None of the statements i)-iii) are correct.

19. Suppose that A is a 2×2 matrix having eigenvalues λ_1 and λ_2 where $\lambda_1 \neq \lambda_2$. Which of the following statements must be true?

i) A is diagonalizable.

ii) if $\mathbf{v}_1$ is an eigenvector for λ_1, and $\mathbf{v}_2$ is an eigenvector for λ_2, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set.

iii) if $\mathbf{v}_1$ is an eigenvector for λ_1, and $\mathbf{v}_2$ is an eigenvector for λ_2, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal set.

A. i) only

B. ii) only

C. iii) only

D. i) and ii) only

E. i) and iii) only

20. Let $L : R^3 \to R^2$ be defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_2 - 2x_3 \end{bmatrix}$. Then the standard matrix representing L is

A. $\begin{bmatrix} 0 & 3 & -2 \\ 1 & 2 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 0 & -2 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -2 & 3 \end{bmatrix}$

$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$[L] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \end{bmatrix}$

21. Let W be the subspace of R^4 spanned by

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}5\\6\\7\\8\end{bmatrix}, \begin{bmatrix}9\\10\\11\\12\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

Then the dimension of the orthogonal complement $W^{\perp}$ is

A. 0

B. 1

C. 2

D. 3

E. 4

$$\text{rank}\begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\\9 & 10 & 11 & 12\\0 & 0 & 0 & 1\end{bmatrix} = 3 \Rightarrow$$

$$\dim W^{\perp} = 4 - \dim W = 1$$

22. Consider the following inconsistent linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix}2 & 1\\1 & 2\\3 & 1\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

The *least squares solution* of the linear system is

A. $\hat{\mathbf{x}} = \begin{bmatrix}1\\1/7\end{bmatrix}$

B. $\hat{\mathbf{x}} = \begin{bmatrix}1/7\\0\end{bmatrix}$

C. $\hat{\mathbf{x}} = \begin{bmatrix}1/7\\1/7\end{bmatrix}$

D. $\hat{\mathbf{x}} = \begin{bmatrix}-1\\1/7\end{bmatrix}$

E. $\hat{\mathbf{x}} = \begin{bmatrix}0\\-1/7\end{bmatrix}$

$$A^TA = \begin{bmatrix}14 & 7\\7 & 6\end{bmatrix} \quad A^Tb = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\left[\begin{array}{cc|c}14 & 7 & 2\\7 & 6 & 1\end{array}\right] \sim \left[\begin{array}{cc|c}7 & 6 & 1\\0 & -5 & 0\end{array}\right]$$

$$\Rightarrow \hat{x} = \begin{bmatrix}\frac{1}{7}\\0\end{bmatrix}$$

23. Let $A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$. For which matrix P is it true that $P^{-1}AP = D$, where D is a diagonal matrix?

A. $P = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}$

B. $P = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}$

C. $P = \begin{bmatrix} -1 & 1 \\ 4 & 1 \end{bmatrix}$

D. $P = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$

E. $P = \begin{bmatrix} 1/3 & 2/3 \\ 1/6 & -1/6 \end{bmatrix}$

$p(\lambda) = \begin{vmatrix} \lambda - 1 & -4 \\ -1 & \lambda + 2 \end{vmatrix}$

$= (\lambda - 1)(\lambda + 2) - 4$

$= \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$

$\lambda_1 = 2$ $\quad \lambda_1 I_2 - A = \begin{bmatrix} 1 & -4 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$ $\quad v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\lambda_2 = -3$

$\lambda_2 I_2 - A = \begin{bmatrix} -4 & -4 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

24. Which of the following statements are always true for a real square matrix A?

i) If A is symmetric, all eigenvalues of A are real.

ii) If A is an orthogonal matrix, then the columns of A form an orthonormal set.

iii) If all eigenvalues of A are 1 then A is similar to the identity matrix.

A. i) only

B. iii) only

C. i) and ii) only

D. i) and iii) only

E. i), ii), and iii)

25. For the differential equation

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x}$$

with initial condition $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, we have $\mathbf{x}(1) =$

A. $(2e^3 + e^{-1}, 2e^3 - e^{-1})^T$
B. $(3e^2 + e^3, e^2 - e^3)^T$
C. $(e^2 - e^3, e^2 + e^3)^T$
D. $(e^4 - 5, e^4 + 5)^T$
E. $(e + e^2, e - e^2)^T$

$$p(\lambda) = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$$

$\lambda_1 = -1$

$$\lambda_1 I - A = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 3$

$$\lambda_2 I - A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X(t) = b_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + b_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad b_1 = 1 \quad b_2 = 2$$

$$X(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} 2e^{3t} - e^{t} \\ 2e^{3t} + e^{t} \end{bmatrix}$$