1. C

This is a separation of variable equation

$$\begin{cases} y\,dy = 3(x+1)^2 dx \\ y(-1) = 2 \end{cases} \Rightarrow \frac{1}{2}y^2 = (x+1)^3 + 2$$

2. D

This is a first order Linear DE $\quad I(x) = e^{\int -\frac{3}{x}dx} = \frac{1}{x^3}$

$$y = x^3 \ln x + Cx^3 \qquad y(1) = 1 \Rightarrow C = 1 \qquad y = x^3\ln x + x^3$$

3. A

This is a homogeneous equation. Let $V = \frac{y}{x}$.

$$xV' = \frac{2V+3}{V^2-1} - V = \frac{-V^3+3V+3}{V^2-1} \Rightarrow -\frac{V^2-1}{V^3-3V-3}dV = \frac{1}{x}dx$$

$$\Rightarrow -\frac{1}{3}\ln(V^3-3V-3) = \ln x + C$$

$$\Rightarrow V^3 - 3V - 3 = Cx^{-3} \Rightarrow 3y^3 - 3x^2y - x^3 = C$$

4. A

$$\frac{dP}{dt} = \frac{k}{P} \Rightarrow P\,dP = k\,dt \Rightarrow P^2 = 2kt + C$$

Plug in $P(0) = 1000$, $P(1) = 2000 \Rightarrow C = 10^6 \quad 2k = 3\times10^6$

5. C

Solve $\begin{cases} vv' = -gR^2/(x+R)^2 \\ v(2R) = v_0 \end{cases} \Rightarrow \frac{1}{2}V^2 = \frac{gR^2}{x+R} + \frac{1}{2}v_0^2 - \frac{gR}{3}$

We want $\frac{gR^2}{x+R} + \frac{1}{2}v_0^2 - \frac{gR}{3} > 0 \Rightarrow V_0^2 > \frac{2Rg}{3} - \frac{2gR^2}{x+R}$

The maximum of the RHS $= \frac{2Rg}{3}$. So $V_0^2 > \frac{2Rg}{3} \Rightarrow V_0 > \sqrt{\frac{2Rg}{3}}$

6. D

$r_1 = 3 \quad c_1 = 0.2 \quad r_2 = 2 \quad V(0) = 50 \quad A(0) = 0$

$$A' = r_1 c_1 - r_2 c_2 = 0.6 - \frac{2A}{V} = 0.6 - \frac{2A}{(r_1 - r_2)t + V_0} = 0.6 - \frac{2A}{50+t}$$

7. C

$\det(-2A) = (-2)^3 \det A = -8 \det A. \quad \det A = 30$

8. B

$\det(-2A^{-1}B) = \det(-2A^{-1}) \det B = -8(\det A)^{-1} \det B = \frac{32}{3}$

9. A

$\det(A^{-1}) = (\det A)^{-1} \quad \det A = -3e^{-t}$

10. E

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 2 & k & 0 \\ k & k & 6 & k+4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & k-1 & 0 \\ 0 & k & 6-k & k+4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{k-1}{2} & 0 \\ 0 & 0 & -k^2-k+12 & 2k+8 \end{array}\right]$$

This system has ∞ many solution if $\begin{cases} -k^2 - k + 12 = 0 & \Rightarrow k = -4 \quad k = 3 \\ 2k + 8 = 0 & \Rightarrow k = -4 \end{cases}$

11. B

A: This is the solution space to an inhomogeneous DE, and this is not a vector space

B. The solution space to a homogeneous DE is a vector space

C — — — — an inhomogeneous system is not a vector space.

12. C.

13 K

$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & 4 & -2 \\ 1 & 5 & k \end{vmatrix} = 0 \Rightarrow k = 8$$

14. C

$A \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \ker T = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\mathrm{Rng}(T) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \end{bmatrix} \right\} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$

15. E

$(D^2+4D+8)^2 y = 0 \Rightarrow ((D+2)^2+4)^2 y = 0 \qquad r = -2 \pm 2i$

$y = c_1 e^{-2t}\cos 2t + c_2 e^{-2t}\sin 2t$

16. B.

Let $Ly = y'' + ay' + by = 0$. $y = c_1 e^x + c_2 e^{2x} \Rightarrow Ly = \underbrace{(D-1)}_{\text{multiplicty } 1}(D-2)y = 0$

The annilator for e^x is $A_1(D) = D-1$

e^{3x} is $A_2(D) = D-3$

1 is $A_3(D) = D$

So a trial solution is $y = c_1 x e^x + c_2 e^{3x} + c_3$

17. A

Similar to #16

18. A

$u_2' = \dfrac{\begin{vmatrix} e^t & 0 \\ e^t & 2(1-t)e^{-t} \end{vmatrix}}{\begin{vmatrix} e^t & t \\ e^t & 1 \end{vmatrix}} = \dfrac{2(1-t)}{e^t(1-t)} = 2e^{-t} \Rightarrow u_2 = -2e^{-t}$

19. D

This is reduction of order method. Plug $y_2 = vx$ into the DE $\Rightarrow$ Answer is D.

20 E.

$y = c_1 e^x + c_2 e^{2x} + 2x + 3$. Use initial condition $\Rightarrow c_1 = 1 \quad c_2 = 0$

21 E

The solution to the Spring-Mass system is $x = 2e^{-2t} - e^{-t}$
This equation passes through the x-axis only once. and $x(0) = 1$.

22. A

$$p(\lambda) = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 2 & 1-\lambda & 4 \\ 0 & -1 & -3-\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} 1-\lambda & 4 \\ -1 & -3-\lambda \end{vmatrix} = -(\lambda+1)^3 \qquad m_{-1} = 3$$

$$A + I \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad d_{-1} = 1$$

23

$$p(\lambda) = \begin{vmatrix} -\lambda & 2 & k \\ 0 & 2-\lambda & k \\ 0 & 2 & k-\lambda \end{vmatrix} = -\lambda^2 (\lambda - k - 2)$$

When $k \neq -2$

$\lambda_1 = 0 \qquad m_1 = 2 \qquad d_1 = 2.$

$$\lambda_2 = -k-2 \qquad \begin{bmatrix} k+2 & 2 & k \\ 0 & -k & k \\ 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} k+2 & 2 & k \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} k+2 & 0 & k+2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$m_2 = d_2 = 1 \Rightarrow$ non-defective

When $k = -2$.

$$\lambda_1 = 0 \qquad m_1 = 3 \qquad A \sim \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad d_1 = 2 \qquad \text{defective}$$

24 D

25. E.

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -e^{-2t} & 0 \\ e^{-2t} & e^{2t} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ e^{t} \end{bmatrix} \xrightarrow{\text{Cramer's rule}} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ -e^{-t} + c_2 \end{bmatrix}$$