1. E

$$\frac{8x^3}{x^4+1}\,dx = \frac{1}{y}\,dy \qquad \text{Let } u = x^4+1$$

$$\int \frac{2\,du}{u} = \int \frac{1}{y}\,dy$$

$$2\ln|u| = 2\ln(x^4+1) = \ln|y| + C$$

$$y = C(x^4+1)^2$$

$$y(0) = C = 2 \quad \Rightarrow \quad y = 2(x^4+1)^2$$

2. B

$$\Phi(x, y) = \int (y\cos xy - \sin x)\,dx$$

$$= \sin(xy) + \cos x + h(y)$$

$$\frac{\partial}{\partial y}\Phi = x\cos(xy) + h'(y) = x\cos(xy)$$

$$\Rightarrow h'(y) = 0 \quad \Rightarrow \quad h(y) = c$$

$$\Phi = \sin(xy) + \cos x + c = 0$$

3. E

This is a first order Linear DE.

$$I(x) = e^{\int -\frac{1}{2x}\,dx} = e^{\ln \frac{1}{\sqrt{x}}} = \frac{1}{\sqrt{x}}$$

$$\left(\frac{1}{\sqrt{x}}\, y\right)' = x \cdot \frac{1}{\sqrt{x}} = \sqrt{x} \quad \Rightarrow \quad \frac{1}{\sqrt{x}}\, y = \int \sqrt{x}\, dx = \frac{2}{3} x^{\frac{3}{2}} + C$$

$$y = \frac{2}{3} x^2 + C\sqrt{x}$$

$$y(1) = \frac{2}{3} + C = \frac{1}{3} \quad \Rightarrow \quad C = -\frac{1}{3}$$

$$y(4) = \frac{2}{3}\, 16 + \left(-\frac{1}{3}\right) 2 = 10$$

4. C

$$A = \begin{bmatrix} 1 & 4 & 2 & 2 \\ 2 & 3 & 0 & 0 \\ 3 & 2 & 0 & 4 \\ 4 & 1 & 2 & 2 \end{bmatrix}$$

We want to find dim (colspace (A))

$$A \sim \begin{bmatrix} 1 & 4 & 2 & 2 \\ 2 & 3 & 0 & 4 \\ 3 & 2 & 0 & 4 \\ 3 & -3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 2 \\ 2 & 3 & 0 & 4 \\ 1 & -1 & 0 & 0 \\ 3 & -3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 2 \\ 2 & 3 & 0 & 4 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & -5 & -4 & 0 \\ 0 & -5 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & -5 & -4 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

dim (colspace (A)) = 3

5. B

$V = 200 \qquad A' + \frac{1}{200} A = 10 \qquad I(x) = e^{\int \frac{1}{200} dt} = e^{\frac{1}{200} t}$

$(e^{\frac{1}{200} t} A)' = 10 e^{\frac{1}{200} t} \quad \Rightarrow \quad e^{\frac{1}{200} t} A = 2000 e^{\frac{1}{200} t} + C$

$A = 2000 + C e^{-\frac{1}{200} t}$

$A(0) = 2000 + C = 0 \Rightarrow C = -2000$

$1000 = 2000 - 2000 e^{-\frac{1}{200} t} \quad \Rightarrow \quad t = 200 \ln 2.$

6. D

This set is linearly dep means that

$$\begin{vmatrix} 2 & 5 & 6 \\ 1 & 3 & 5 \\ a & 7 & 7 \end{vmatrix} = 0$$

Computing this determinant, we obtain

$$\begin{vmatrix} 2 & 5 & 6 \\ 1 & 3 & 5 \\ a & 7 & 7 \end{vmatrix} = 42 + 42 + 25a - 18a - 70 - 35 = 7a - 21 = 0$$

$\Rightarrow a = 3.$

7. C

We want

$$0 \neq \begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & k-1 & 0 & 0 \\ 3k & 4 & 2 & k \\ 9 & -8 & 1 & 4 \end{vmatrix} = 2 \begin{vmatrix} k-1 & 0 & 0 \\ 4 & 2 & k \\ -8 & 1 & 4 \end{vmatrix} = 2(k-1) \begin{vmatrix} 2 & k \\ 1 & 4 \end{vmatrix} = 2(k-1)(8-k)$$

$k \neq 1, 8$

8. A

This means

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 5 \\ 2 & 0 & 2 & 6k \\ 2 & 2 & 0 & -2 \\ 1 & 1 & 3 & 2 \end{array}\right] = [A \mid b] \text{ is consistent}$$

$$[A \mid b] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -1 & 2 & 5 \\ 2 & 2 & 0 & -2 \\ 2 & 0 & 2 & 6k \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -2 & -5 \\ 2 & 2 & 0 & -2 \\ 0 & -2 & 2 & 6k+2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & -2 & 2 & 6k+2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & 6k \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 6k-6 \end{array}\right]$$

This system is consistent if and only if $6k - 6 = 0 \Rightarrow k = \underline{1}$.

9. B.

Note that if we expand this determinant along the first column, x^3 terms only appears in

$$-x^2 \begin{vmatrix} 8 & 7 & 1 \\ x & 7 & -2 \\ 1 & 4 & 7 \end{vmatrix} = -x^2 \det B$$

Again, expanding along the first column of B, we get x^3 terms only in

$$-x^2(-x) \begin{vmatrix} 7 & 1 \\ 4 & 7 \end{vmatrix} = 45x^3$$

10. D

$\det A = (1+i)i + 1 = i - 1 + 1 = i$

$$A^{-1} = \frac{1}{i}\underbrace{\begin{bmatrix} i & 1 \\ -1 & 1+i \end{bmatrix}}_{\text{adjoint of } A} = -i\begin{bmatrix} i & 1 \\ -1 & 1+i \end{bmatrix} = \begin{bmatrix} 1 & -i \\ i & 1-i \end{bmatrix}$$

11. B

We plug y_2 into the DE.

$$(x+1)[v e^x]$$
$$-2x\,[v e^x + v' e^x]$$
$$+(x-1)[v e^x + 2v' e^x + v'' e^x] = 0$$

$$2(x-1)v' e^x + (x-1)v'' e^x - 2x v' e^x = 0$$

$(x-1)v'' - 2v' = 0$ Let $w = v'$

$(x-1)w' - 2w = 0 \Rightarrow \frac{dw}{w} = \frac{2}{x-1}dx$

$$w = C_1 (x-1)^2$$

$$v = \int C_1 (x-1)^2 dx = \frac{1}{3}C_1 (x-1)^3 + C_2$$

$$y = C_1 \underbrace{\left[\tfrac{1}{3}(x-1)^3\right]e^x}_{y_2} + C_2 \underbrace{e^x}_{y_1}$$

12. A

(i) Consider $\begin{cases} 0 \cdot x_1 = 1 \\ 0 \cdot x_1 = 2. \end{cases}$

(ii) Consider $0 \cdot x_1 + 0 \cdot x_2 = 1.$

(iii) and (iv) The same as (ii).

13. C

We map P_2 to $\mathbb{R}^3$ by

$$a + bx + cx^2 \to \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

We want.

$$0 \neq \begin{vmatrix} k & 3 & 2 \\ 0 & k & 1 \\ -k & 0 & k \end{vmatrix} = k \begin{vmatrix} k & 1 \\ 0 & k \end{vmatrix} - k \begin{vmatrix} 3 & 2 \\ k & 1 \end{vmatrix}$$

$$= k^3 - k(3 - 2k) = k(k^2 + 2k - 3) = k(k+3)(k-1)$$

14 E.

We take the standard basis $\{1, x, x^2, x^3\}$ for P_3

We map P_3 to $\mathbb{R}^4$ by $a + bx + cx^2 + dx^3 \to [a\ b\ c\ d]^T$

Then the matrix representation of T is

$$[T] = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} T(1) & T(x) & T(x^2) & T(x^3) \end{bmatrix}.$$

$\det [T] = 1 \neq 0.$

$\dim(\mathrm{Rng}(T)) = 4.$

15. D

By Section 5.7 # 29 (c)

Sum of eigenvalues of A = $\operatorname{tr} A = 0+3+4 = 7$

16. B

$$\begin{bmatrix} 1 & 5 & 4 & 3 & 2 \\ 1 & 6 & 6 & 6 & 6 \\ 1 & 7 & 8 & 10 & 12 \\ 1 & 6 & 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 4 & 3 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 7 & 10 \\ 0 & 1 & 2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -6 & -12 & -18 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -6 & 0 & 6 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad X = s\begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -6 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

17. E

$\det A = 1 \cdot 1 \cdot 3 = 3 \neq 0$. So (i), (ii) (iii) are true.

Since 2 is not an eigenvalue, $\det(A - 2I) \neq 0$. So (iv) holds.

18. C

Since $\lambda_1 = 2$ has multiplicity 2, most likely its eigenspace will have dimension one. So we try $\lambda_1 = 2$ first

$$A - 2I \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad X = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

19. D

$(D^2+4)(D-2)(D+2)y=0$

$y = C_1\cos 2t + C_2\sin 2t + C_3e^{2t} + C_4e^{-2t}$

$y(0) = C_1 + C_3 + C_4 = 1$

$y(x) \to 0$ as $x \to \infty$ $\Rightarrow C_1 = C_2 = C_3 = 0$.

So $C_4 = 1$ and $y = e^{-2t}$

$y''(0) = 4e^{2t}\Big|_{t=0} = 4$

20. E.

$$U_2'(x) = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\sin x}{1} \Rightarrow U_2(x) = \int \sin x\, dx = -\cos x$$

21. C

The general solution to $y''+9y=0$ is $C_1\sin(3x) + C_2\cos(3X)$

The annihilator is D^2+16.

So a trial solution is $y_T = A\sin(4x) + B\cos(4x)$.

Plug into the DE. We have

$$-7A\sin(4x) - 7B\cos(4x) = 7\cos(4x)$$

$\Rightarrow A=0 \quad B=-1$

$$y = C_1\sin(3x) + C_2\cos(3x) - \cos(4x)$$

22. D

$r_1 = -1 \qquad e^{-x}$

$r_2 = 3 \qquad e^{3x} \qquad x e^{3x}$

$r_{3,4} = 3 \pm 2i \qquad e^{3x}\cos 2x \qquad e^{3x}\sin 2x$

$$y = c_1 e^{-x} + c_2 e^{3x} + c_3 x e^{3x} + e^{3x}(c_4 \cos 2x + c_5 \sin 2x)$$

23. A

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$

$\lambda_1 = 3 \qquad A - 3I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1 \qquad A + I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$X(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow c_1 = 2 \qquad c_2 = -1$$

$$X(t) = 2e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad x_1(1) = 2e^3 + e^{-1}$$

24. D

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$$e^{\lambda_1 t} v_1 = e^{3t}(\cos 2t - i\sin 2t)\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$$

$$= e^{3t}\left\{\underbrace{\left(\cos 2t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin 2t\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)}_{\begin{bmatrix} \cos 2t \\ 2\sin 2t \end{bmatrix}} + i\underbrace{\left(\cos 2t\begin{bmatrix} 0 \\ 2 \end{bmatrix} - \sin 2t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)}_{\begin{bmatrix} -\sin 2t \\ 2\cos 2t \end{bmatrix}}\right\}$$

The general solution is

$$v = c_1 e^{3t}\begin{bmatrix} \cos 2t \\ 2\sin 2t \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} -\sin 2t \\ 2\cos 2t \end{bmatrix}$$

25 B

$$u_1' = \frac{\begin{vmatrix} e^t & e^{-t} \\ -e^t & 3e^{-t} \end{vmatrix}}{\begin{vmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{vmatrix}} = \frac{4}{2} = 2 \Rightarrow u_1 = 2t.$$