MA26500: EXAM I NAME: ____________________

1. A is an $m \times n$ matrix and b is an $m \times 1$ vector. The equation $AX = b$ has infinitely many solutions. Consider the following statements:

(i) $m \leq n$

(ii) $n \leq m$

(iii) $\text{rank}(A) = n$

(iv) $\text{rank}(A) < n$

(v) $\det(A) = 0$

Which must be true?

A. only (i) and (v)

B. only (iv)

C. only (v)

D. only (iii) and (v)

E. None of the above.

(i) Counterexample.

$$\begin{cases} x + y = 1 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

(ii) & (iv) Counterexample:

$$x + y = 1.$$

(v) If $m \neq n$, we cannot define $\det(A)$.

2. Only one of the following is NOT always true. Which one is it?

A. The product of two 3×3 diagonal matrices is a diagonal matrix.

B. For any two $n \times n$ matrices A, B, $(A+B)^2 = A^2 + AB + BA + B^2$.

C. For any two matrices A, B, if $AB = 0$ then either $A = 0$ or $B = 0$.

D. Product of two 3×3 upper triangular matrices is an upper triangular matrix.

E. The transpose of a symmetric matrix is symmetric.

For example : $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

3. If $A = \begin{bmatrix} 0 & 2 & -2 \\ 2 & 3 & 0 \\ 1 & 5 & -3 \end{bmatrix}$. What is the **sum** of the entries in the third row of A^{-1}?

A. $-\frac{5}{2}$

B. $\frac{5}{2}$

C. $\frac{3}{2}$

D. 1

E. 5

$$|A| = \begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 3 \\ 1 & 5 & 2 \end{vmatrix} = -2 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -2$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7$$

$$A_{23} = -\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 2$$

$$A_{33} = \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} = -4$$

$$a_{31}^{-1} + a_{32}^{-1} + a_{33}^{-1} = \frac{7+2-4}{-2} = -\frac{5}{2}$$

4. Which of the following are true?

(i) An $n \times n$ elementary matrix is always nonsingular.

(ii) If A is nonsingular and diagonal, then A^{-1} is also nonsingular and diagonal.

(iii) An $n \times n$ matrix is nonsingular if and only if its reduced row echelon form is I_n.

(iv) If A is symmetric, then adj$A$ is also symmetric.

A. only (i) and (ii)

B. (i), (ii), (iii)

C. (i), (ii), (iv)

D. (ii), (iii), (iv)

E. (i), (ii), (iii), (iv)

(iv) $AA^{-1} = I_n \Rightarrow (A^{-1})^T A^T = I_n$

$A \overset{\text{sym}}{\Rightarrow} (A^{-1})^T A = I_n \Rightarrow (A^{-1})^T = A^{-1}$

$\Rightarrow A^{-1}$ sym

$\text{adj} A = |A| A^{-1}$ sym.

5. Let A be an invertible matrix with the inverse

$$A^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

Which of the following statement is NOT always true?

(i) For arbitrary 2×2 matrices B and C. If $AB = AC$, then $B = C$.

(ii) A^T is invertible.

(iii) For arbitrary 2×2 matrices B and C. If $BA = CA$, then $B = C$.

(iv) rank(A)=2

(v) A is symmetric.

A. (i) and (iii)

B. (ii) and (v)

C. (i), (ii), (iii) and (v)

D. (i), (iii) and (v)

E. None of the above.

6. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & 1 & 0 \\ -1 & -4 & 1 & -1 \\ 1 & 2 & 0 & -4 \end{bmatrix}$$

via **the row echelon form method.**

$$[A \mid I_4] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -3 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & -1 & -4 & 0 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 0 & -1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & -1 & 0 & -8 & 11 & 4 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 0 & -1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 8 & -11 & -4 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & -17 & 25 & 9 & 2 \\ 0 & 1 & 0 & 0 & 9 & -13 & -5 & -1 \\ 0 & 0 & 1 & 0 & 8 & -11 & -4 & -1 \\ 0 & 0 & 0 & 1 & -2 & 3 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{c|cccc} & -26 & 38 & 14 & 3 \\ I_4 & 9 & -13 & -5 & -1 \\ & 8 & -11 & -4 & -1 \\ & -2 & 3 & 1 & 0 \end{array}\right]$$

7. For two $n \times n$ matrices A and B, how many of the following statements are true.

(a) $\det(AB) = \det(A)\det(B)$

(b) $\det(A) = \det(A^T)$

(c) For $k \neq 0$, $\det(kA) = k\det(A)$.

(d) $\det(A^{-1}) = \det(A)^{-1}$

(e) If $A = PBP^{-1}$ for an invertible matrix P, then $\det(A) = \det(B)$.

A. 1

B. 2

C. 3

D. 4

E. 5

8. For what α, the system of linear equations

$$\begin{cases} 2x + 5y + (3\alpha)z + 4w &= 0 \\ (\alpha - 1)y + 4z - 3w &= 0 \\ 2z + w &= 0 \\ (\alpha)z + 4w &= 0 \end{cases}$$

has non-trivial solutions?

A. $\alpha = 0, 2$

B. $\alpha = 1, 5$

C. $\alpha = -1, -5$

D. $\alpha = 1, 8$

E. $\alpha = 0, 1$

$$\begin{bmatrix} 2 & 5 & 3\alpha & 4 \\ 0 & \alpha - 1 & 4 & -3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & \alpha & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & 3\alpha & 4 \\ 0 & \alpha - 1 & 4 & -3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 - \frac{\alpha}{2} \end{bmatrix} = B$$

When $\alpha = 8$ $\quad |B| = 0 \Rightarrow$ there is no-trivial sol.

When $\alpha = 1$ $\quad |B| = 0 \Rightarrow$ - - - — — -

When $\alpha \neq 1, 8$ $\quad |B| \neq 0 \Rightarrow$ only trivial sol.

9. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and

$$AX = \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}$$

What is x_2?

A. 11

B. 20/3

C. 5/4

D. 1

E. 0

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & -3 & -5 \end{vmatrix} = 3$$

$$X_2 = \frac{\begin{vmatrix} 1 & -1 & 3 \\ 2 & 2 & 2 \\ 2 & -4 & 1 \end{vmatrix}}{3} = -\frac{28}{3}$$

10. Which of the following sets are vector space?

- W_1= polynomials $P(t) = at^3 + bt^2 + ct + d$ satisfying $a - b + c = d$ with usual addition and scalar multiplication.
- W_2= degree ≤ 5 polynomials $P(t)$ satisfying $P(2) = 0$ with usual addition and scalar multiplication.
- W_3= vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ satisfying $x+y=0$ with usual addition and scalar multiplication.
- W_4= all solutions of the linear system $Ax = 0$ for A an $m \times n$ matrix with usual addition and scalar multiplication.
- W_5= 3×3 upper triangular matrices with usual addition and scalar multiplication.

A. W_1, W_2, W_3, W_3

B. W_1, W_2, W_4, W_5

C. W_2, W_3, W_4, W_5

D. W_1, W_3, W_4, W_5

E. W_1, W_2, W_3, W_4, W_5

11. Determine which one of the following expressions is the general solution to the homogeneous system of equations

$$\begin{cases} x_1 + 5x_2 + 4x_3 + 3x_4 + 2x_5 & = 0 \\ x_1 + 6x_2 + 6x_3 + 6x_4 + 6x_5 & = 0 \\ x_1 + 7x_2 + 8x_3 + 10x_4 + 12x_5 & = 0 \\ x_1 + 6x_2 + 6x_3 + 7x_4 + 8x_5 & = 0 \end{cases}$$

A. $s\begin{bmatrix}1\\0\\-6\\0\\6\end{bmatrix}$

B. $s\begin{bmatrix}6\\-2\\1\\0\\0\end{bmatrix}+t\begin{bmatrix}-6\\2\\0\\-2\\1\end{bmatrix}$

C. $s\begin{bmatrix}0\\-1\\-1\\0\\6\end{bmatrix}$

D. $s\begin{bmatrix}1\\1\\0\\-2\\1\end{bmatrix}+t\begin{bmatrix}0\\-4\\-1\\7\\0\end{bmatrix}$

E. $s\begin{bmatrix}1\\1\\0\\-2\\1\end{bmatrix}+t\begin{bmatrix}0\\-4\\-1\\7\\0\end{bmatrix}+r\begin{bmatrix}1\\-3\\-1\\5\\1\end{bmatrix}$

$$\begin{bmatrix}1&5&4&3&2\\1&6&6&6&6\\1&7&8&10&12\\1&6&6&7&8\end{bmatrix} \sim \begin{bmatrix}1&0&-6&0&6\\0&1&2&0&-2\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}$$

free variables

Set $X_3 = s$, $X_5 = t$ $\Rightarrow$ $X_1 = 6s - 6t$, $X_2 = -2s + 2t$, $X_4 = -2t$

$$\begin{bmatrix}X_1\\ \vdots\\ X_5\end{bmatrix} = \begin{bmatrix}6s-6t\\-2s+2t\\s\\-2t\\t\end{bmatrix} = s\begin{bmatrix}-6\\-2\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-6\\2\\0\\-2\\1\end{bmatrix}$$