1. Determine all values of k so that the vectors $(2, -k, 1)$, $(1, -1, 1)$, $(0, 1, -k)$ are linearly dependent.

A. $k \neq 1$

B. $k \neq 2$

C. $k \neq -1$

D. $k = -1$

E. $k = 1$

answer: E

$$0 = \begin{vmatrix} 2 & 1 & 0 \\ -k & -1 & 1 \\ 1 & 1 & -k \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 2-k & -1 & 1 \\ -1 & 1 & -k \end{vmatrix}$$

$$= -1 \begin{vmatrix} 2-k & 1 \\ -1 & -k \end{vmatrix} = -(k^2 - 2k + 1)$$

$$= -(k-1)^2 \Rightarrow k = 1$$

2. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{bmatrix}$. Which of the following statements are true?

(i). The first two columns form a basis for the column space of A.

(ii). The vector $\begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$ is in the column space of A.

(iii). The rows are linearly independent.

A. (i) only

B. (iii) only

C. (i) and (ii) only.

D. (ii) and (iii) only.

E. All of them.

answer: A

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 7 & -4 \\ 7 & 8 & 9 & 10 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -3 & -6 & -9 & -4 \\ 0 & -6 & -12 & -18 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 4/3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$ no solution

columns containing leading 1's.

The first two columns are L. I.

rank $A = 2$. So the rows are L. D.

3. Determine all values of k so that $\{k - kx^2, 3 + kx, 2 + x + kx^2\}$ is a basis for P_2, the vector space of all polynomials of degree ≤ 2.

A. $k = 0, 3, 4$

B. $k \neq 0, 3, 4$

C. $k \neq -3, 0, 1$

D. $k = -3, 0, 1$

E. $k \neq 1$

Answer: C

$$ax^2 + bx + c \to \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$0 \neq \det \begin{bmatrix} -k & 0 & k \\ 0 & k & 1 \\ k & 3 & 2 \end{bmatrix} = \begin{vmatrix} 0 & 0 & k \\ 1 & k & 1 \\ k+2 & 3 & 2 \end{vmatrix}$$

$$= k \begin{vmatrix} 1 & k \\ k+2 & 3 \end{vmatrix} = k(3 - k^2 - 2k) = -k(k+3)(k-1)$$

$k \neq -3, 0, 1$

4. Suppose that A is a real 3×5 matrix such that the $\text{rank}(A) = 3$. Consider the following statements:

(i) $\text{rank}(A^T) = 5$

(ii) The nullity of $A = 2$.

(iii) The row space of A is $\mathbb{R}^5$.

(iv) The column space of A is $\mathbb{R}^3$.

(v) $AX = 0$ has non-trivial solution.

which of these statements **must** be true

A. only (i) and (v)

B. only (ii) and (iii)

C. only (ii) and (iv)

D. only (ii), (iv) and (v).

E. All of them have to be true.

answer: D

(i) $\text{rank}(A^T) = \text{rank } A = 3$

(ii) nullity of $A = 5 - \text{rank}(A) = 2$

(iii) $\dim(\text{rowspace}(A)) = 3$

(iv) $\dim(\text{colspace}(A)) = 3 \Rightarrow \text{colspace}(A) = \mathbb{R}^3$

(v) null space of (A) is 2-dimensional.

5. Which of the following are linear transformations?

(i) $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L(x, y, z) = (y + z + 1, 2x)$.

(ii) $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(x, y, z) = 5(x, y, z)$.

(iii) $L : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $L(x, y) = (-3y, x^2, x + y)$.

A. (i) only.

B. (i) and (ii) only.

C. (ii) and (iii) only.

D. (ii) only.

E. None of the above.

answer: D

(i) L is not a L.T. since $L\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

(iii) L is not a L.T. since

$$L\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix}$$

$$L\begin{pmatrix}3\\3\end{pmatrix} = \begin{pmatrix}-9\\9\\6\end{pmatrix} \neq 3L\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-9\\3\\6\end{pmatrix}$$

6. Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, then the projection of $\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ onto W is

$= u$

A. $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}$

C. $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

D. $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$

E. $\begin{bmatrix} -1/6 \\ 1/2 \\ 1/6 \end{bmatrix}$

answer: C

Let $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Use Gram-Schmidt to find an orthogonal basis $\{v_1 \ v_2\}$

Step1: $v_1 = u_1$

Step 2: $v_2 = u_2 - \dfrac{(u_2, v_1)}{\|v_1\|^2} v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \dfrac{1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$\text{Proj}_W u = \frac{(u, v_1)}{\|v_1\|^2} v_1 + \frac{(u, v_2)}{\|v_2\|^2} v_2$$

$$= \frac{-1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{4}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

7. Let $C[-1,1]$ be the vector space of all real-valued continuous functions on $[0,1]$. Define an inner product on $C[-1,1]$ by

$$(f,g) = \int_{-1}^{1} f(x)g(x)\,dx.$$

Then which of the following set functions are orthogonal.

A. $1, e^x, e^{-x}$.

B. x, x^2, x^3.

C. $1, \sin x, \cos x$

D. $1, x, 3x^2 - 1$

E. None of the above.

answer: D

Note that if $f \cdot g \geq 0$ or $f \cdot g \leq 0$ $(fg \not\equiv 0)$

then $(f, g) \neq 0$

A. all the functions are positive

B. $x \cdot x^3 = x^4 \geq 0$ $(x^4 = 0$ only at $0)$

C $1 \cdot \cos x > 0$

D $1 \cdot x$ and $x \cdot (3x^2 - 1)$ are odd functions. Their integrals over $[-1, 1] = 0$

$$\int_{-1}^{1} 1 \cdot (3x^2 - 1)\,dx = x^3 - x \Big|_{-1}^{1} = 0$$

This is an orthogonal set

8. Which of the following set of vectors forms a basis for $\mathbb{R}^3$?

A. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ × the last row is zero.

B. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \right\}$ × contains zero vector.

E. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}$ × too many vector

answer: B

B $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 6$

C $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$ the first & third row are zero.

9. Let $W = \text{Span}\{\begin{bmatrix}1\\3\\-1\\7\end{bmatrix}, \begin{bmatrix}5\\1\\5\\7\end{bmatrix}, \begin{bmatrix}7\\0\\8\\7\end{bmatrix}\}$. What is the dimension of $W^{\perp}$?

A. 0

B. 1

C. 2

D. 3

E. 4

answer: C

$$A = \begin{bmatrix}1 & 5 & 7\\3 & 1 & 0\\-1 & 5 & 8\\7 & 7 & 7\end{bmatrix} \sim \begin{bmatrix}1 & 5 & 7\\0 & -14 & -21\\0 & 10 & 15\\0 & -28 & -42\end{bmatrix} \sim \begin{bmatrix}1 & 5 & 7\\0 & 2 & 3\\0 & 2 & 3\\0 & 2 & 3\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & 5 & 7\\0 & 2 & 3\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

rank $A = 2 \Rightarrow W =$ colspace (A) is of dimension 2

$\dim W^{\perp} = 4 - \dim W = 2$

10. Suppose that $W =$ the plane $x + 2y - 3z = 0$ and $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. What is the distance from v to W?

A. 0

B. 4

C. $\sqrt{4}$

D. 1

E. $\sqrt{2}$

answer: A

$1 + 2 \cdot 1 - 3 \cdot 1 = 0 \Rightarrow v \in W$

$\Rightarrow$ distance of v to $W = 0$.