EXAMPLES OF SECTION 1.4, 1.5

Example 1. *Assume that* A, B *are* $m \times n$ *matrices and* C *is* $n \times p$*. Prove that* $(A+B)C = AC + BC$*.*

Proof. Let $(A+B)C = D = [d_{ij}]$. Then

$$\begin{aligned} d_{ij} &= \sum_{k=1}^{n} (A+B)_{ik} c_{kj} = \sum_{k=1}^{n} (a_{ik} + b_{ik}) c_{kj} \\ &= \sum_{k=1}^{n} a_{ik} c_{kj} + \sum_{k=1}^{n} b_{ik} c_{kj} = (AC)_{ij} + (BC)_{ij} \\ &= (AC + BC)_{ij}. \end{aligned}$$

By the definition of matrix equality, this implies $(A+B)C = AC + BC$. □

Example 2. *Prove that the product of any two upper(lower) triangular matrices is again a(n) upper(lower) triangular matrix.*

Remark 3. *For any* $n \times n$ *matrix* A*,* A *is upper triangular if and only if*

$$a_{ij} = 0, \quad \textit{for all } i > j,$$

and A *is lower triangular if and only if*

$$a_{ij} = 0, \quad \textit{for all } i < j.$$

Proof. (of Example 2) We will prove this assertion for upper triangular matrices only. The proof for lower triangular case is exactly the same.

Suppose that $A_{n \times n}, B_{n \times n}$ are both upper triangular. Let $AB = C = [c_{ij}]$. Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

We consider only c_{ij} with

$$i > j. \tag{1}$$

Case I: $k > j$

In this case, $b_{kj} = 0$. So $a_{ik} b_{kj} = 0$.

Case II: $k \leq j$.

In this case, by (1)

$$i > j \geq k,$$

which implies $a_{ik} = 0$. Thus $a_{ik} b_{kj} = 0$.

In both cases, we obtain

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = 0, \quad \text{for } i > j.$$

We thus infer from Remark 3 that C is upper triangular. □