EXAMPLES OF SECTIONS 6.1

Question 1. T is a linear transformation from $\mathbb{P}_2$ to $\mathbb{P}_2$, $\mathbb{P}_2$ is the space of all polynomials of degree no more than 2, and

$$T(x^2-1) = x^2+x-3, \quad T(2x) = 4x, \quad T(3x+2) = 2x+6.$$

Find $T(1)$, $T(x)$, and $T(x^2)$.

Question 2. Prove that $T : C^k(I) \to C^{k-1}(I)$: $f \mapsto \frac{d}{dx} f$ is a liner transformation.

Question 3. Let $I = [a,b]$. Prove that $T : C(I) \to \mathbb{R}$: $f \mapsto \int_a^b f(x)\,dx$ is a liner transformation.

Solutions.

1. We identify T as a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ by the map

$$ax^2+bx+c \mapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

By the given conditions, we have

$$T(1,0,-1) = (1,1,-3), \quad T(0,2,0) = (0,4,0), \quad T(0,3,2) = (0,2,6).$$

We immediately have

$$T(0,1,0) = \frac{1}{2} T(0,2,0) = (0,2,0).$$

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix}.$$

Solving

$$A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

we get $\vec{x} = \begin{bmatrix} 0 \\ -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$. Therefore,

$$T(0,0,1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 2 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}.$$

Finally,

$$T(1,0,0) = T(1,0,-1) + T(0,0,1) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Now we restore this result back to the space $\mathbb{P}_2$ and obtain

$$T(1) = -2x + 3, \quad T(X) = 2x, \quad T(x^2) = x^2 - x.$$

2. We will verify the conditions of linear transformation in one step:

$$\begin{aligned} T(c_1 f_1 + c_2 f_2) &= \frac{d}{dx}(c_1 f_1 + c_2 f_2) = c_1 \frac{d}{dx} f_1 + c_2 \frac{d}{dx} f_2 \\ &= c_1 T(f_1) + c_2 T(f_2). \end{aligned}$$

Therefore, T is a liner transformation.

Remark 0.1. *This is a very important example of linear transformation. We shall refer back to this example in Chapter 8.*

3. *We verify two steps in one.*

$$\begin{aligned} T(c_1 f_1 + c_2 f_2) &= \int_a^b (c_1 f_1(x) + c_2 f_2(x))\, dx = c_1 \int_a^b f_1(x)\, dx + c_2 \int_a^b f_2(x)\, dx \\ &= c_1 T(f_1) + c_2 T(f_2). \end{aligned}$$