MATRICES AND VECTOR SPACES REVIEW

Suppose that A is an $n \times n$ matrix. Then the followings are equivalent:

- A is invertible.
- There is an $n \times n$ matrix B such that $BA = I_n$.
- There is an $n \times n$ matrix B such that $AB = I_n$.
- A^T is invertible.
- rank(A)=n=dim(colspace(A))=dim(rowspace(A)).
- $Ax = 0$ has only trivial solution.
- The null space of A is $\{0\}$.
- For any b in $\mathbb{R}^n$, $Ax = b$ has a unique solution.
- ref(A) is an upper triangular matrix with identical 1 on the main diagonal.
- rref(A)=I_n.
- The columns of A are linearly independent.
- The rows of A are linearly independent.
- The columns of A form a spanning set of $\mathbb{R}^n$.
- The rows of A form a spanning set of $\mathbb{R}^n$.
- The columns of A form a basis for $\mathbb{R}^n$.
- The rows of A form a basis for $\mathbb{R}^n$.
- colspace(A)=rowspace(A)=$\mathbb{R}^n$.
- If $\{v_1, \cdots, v_n\}$ is a basis (viewed as column vectors) for $\mathbb{R}^n$, then $\{Av_1, \cdots, Av_n\}$ is again a basis for $\mathbb{R}^n$.
- $\det A \neq 0$.
- All eigenvalues of A are nonzero.

Suppose that A is an $m \times n$ matrix, not necessarily square. Then the followings are true.

- rank(A) $\leq \min\{m, n\}$.

- If $m < n$, then
  - $Ax = 0$ must have non-trivial solution;
  - the null space of A is non-trivial.
- If $m > n$, then
  - $Ax = b$ is not consistent for all b in $\mathbb{R}^m$;
  - the columns of A cannot be a spanning set of $\mathbb{R}^m$;
  - $\text{colspace}(A) \neq \mathbb{R}^m$.
- If $\text{rank}(A) = m$, then
  - $m \leq n$
  - $Ax = b$ must have at least one solution;
  - $\text{rowspace}(A)$ has dimension m;
  - the rows of A are linearly independent;
  - $\text{colspace}(A)$ has dimension m;
  - the null space of A has dimension $n - m$.
- If $\text{rank}(A) = n$, then
  - $n \leq m$
  - $Ax = b$ either has no solution or only one solution;
  - $\text{rowspace}(A)$ has dimension n;
  - $\text{colspace}(A)$ has dimension n;
  - the columns of A are linearly independent;
  - the null space of A has dimension 0;