# CHAPTER 4 REVIEW

## 1. Finite dimensional vector spaces

Any finite dimensional vector space can be identified as a Euclidean space.

**Example 1.1.** *$M_{m\times n}(\mathbb{R}) = M_{mn}(\mathbb{R})$, the space of all real valued $m \times n$ matrix, can be identified as $\mathbb{R}^{mn}$. Every matrix*

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

*is mapped to the column vector*

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} & a_{21} & \cdots & a_{2n} & \cdots & a_{m1} & \cdots & a_{mn} \end{bmatrix}^T.$$

**Question:** How to find a basis for $M_{m\times n}(\mathbb{R})$?
**Answer:** $\{M_{ij} : 1 \le i \le m, 1 \le j \le n\}$ forms a basis for $M_{m\times n}(\mathbb{R})$, where $M_{ij}$ is the $m \times n$ matrix with 1 in the $(i, j)$-entry and 0 elsewhere.

**Example 1.2.** *$S_n(\mathbb{R})$, the space of all real valued symmetric $n \times n$ matrix, can be identified as $\mathbb{R}^{\frac{n(n+1)}{2}}$, thus has dimension $\frac{n(n+1)}{2}$ (Consider why?). Consider how to find a basis for $S_n(\mathbb{R})$. Recall any symmetric matrix is symmetric with respect to the main diagonal.*

**Example 1.3.** *$\mathbb{P}_n$, the space of all polynomials of degree no more than $n$, can be identified as $\mathbb{R}^{n+1}$. Every polynomial*

$$a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$$

*is mapped to the column vector*

$$\begin{bmatrix} a_0 & \cdots & a_n \end{bmatrix}^T.$$

**Question:** How to find a basis for $\mathbb{P}_n$?
**Answer:** $\{t^n, t^{n-1}, \cdots, t, 1\}$ forms a basis for $\mathbb{P}_n$.

**Example 1.4.** *More generally, any $n$-dimensional vector space $V$ can be identified as $\mathbb{R}^n$. Since $V$ is $n$-dimensional, we can find a a basis $\{v_1, \cdots, v_n\}$. For every $v$ in $V$, we can find a unique linear combination*

$$v = c_1 v_1 + \cdots + c_n v_n.$$

*(Consider why this linear combination is unique). Then the vector $v$ is mapped to the column vector*

$$\begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}^T.$$

*This column vector is called the coordinates of $v$ with respect to the basis $\{v_1, \cdots, v_n\}$.*

## 2. Infinitely dimensional vector spaces

There does exist infinitely dimensional vector space. A vector space is of infinite dimension if it has a basis containing infinitely many vectors.

**Example 2.1.** *$P :=$ the set of all polynomials is an infinite dimensional vector space. $\{1, x, x^2, \cdots\}$ is a basis of $P$. This space can be recognized as $\mathbb{R}^\infty$.*

**Example 2.2.** *Let $I$ be an interval or the real line $\mathbb{R}$.*

$$\begin{aligned} C^n(I) = \{f : I \to \mathbb{R} : f \ n \text{ times differentiable}, \\ f, f', \cdots, f^{(n)} \text{ are all continuous.}\} \end{aligned}$$

*is an infinite dimensional vector space. Indeed, there is a basis of $C^n(I)$ containing $\{1, x, x^2, \cdots\}$, and thus has infinitely many elements.*

## 3. Subspaces

A subspace $S$ is a subset of a vector space $V$, which is a vector space itself if equipped with the vector addition and scalar multiplication of $V$.

**Example 3.1.** *State all the subspaces of $\mathbb{R}^3$.*

**Solution.** Subspaces of $\mathbb{R}^3$ are $\mathbb{R}^3$ itself and all the planes and lines passing through the origin. ◀

**Remark 3.2.** *A subset $S$ of a vector space $V$ is a subspace if and only if $S$ is closed under the* **same** *vector addition and scalar multiplication.*

**Remark 3.3.** *If $0_V$ is not in $S$, then $S$ is not a subspace.*

**Example 3.4.** *The null space of an $m \times n$ matrix $A$, that is, the set of solutions to the homogeneous linear system*

$$Ax = 0, \tag{1}$$

*is a subspace of $\mathbb{R}^n$. The dimension of this subspace is $n - \text{rank}(A)$.*

*Proof.* If the $n$-column vectors $x, y$ are both solutions to (1). Then

$$Ax + Ay = A(x + y) = 0,$$

which implies that $x + y$ is again a solution. For any scalar $c$,

$$cAx = A(cx) = 0,$$

which implies that $x + y$ is again a solution. $\square$

**Example 3.5.** *The set of all the polynomial $ax^2+bx+c$ satisfying $a+b=c$ forms a subspace of $\mathbb{P}_2$. The dimension of this subspace is 2.(Consider why?)*

## 4. Linear dependence and independence

**Proposition 4.1.** *A set of vectors $S = \{v_1, \cdots, v_n\}$ is linearly dependent if and only if some $v_k$ can be expressed as a linear combination of the* **other** *vectors in $S$.*

**Remark 4.2.** *In the textbook, the author proved that $S = \{v_1, \cdots, v_n\}$ is linearly dependent if and only if some $v_k$ can be expressed as a linear combination of the proceeding vectors in $S$, i.e., $v_k = c_1v_1 + \cdots + c_{k-1}v_{k-1}$.*

**Summary:** How to determine whether a given set $\{v_1, \cdots, v_n\}$ in $\mathbb{R}^m$ is linearly independent or not?

- If $m < n$, not linearly independent.
- If $m \geq n$, let $A = [v_1 \quad \cdots \quad v_n]$
- If $\text{rank}(A) = n$, that is, the number of columns of $A$, then $\{v_1, \cdots, v_n\}$ is linearly independent. Otherwise, not.

**Remark 4.3.** *When $m = n$, $A$ is a square matrix. In this case, the last step can be replaced by computing the determinant of $A$. More precisely, if $\det(A) \neq 0$, then $\{v_1, \cdots, v_n\}$ is linearly independent. Otherwise, not.*

**Summary:** How to find a linearly independent subset out of a given subset $\{v_1, \cdots, v_n\}$ in $\mathbb{R}^m$?

- Let $A = [v_1 \quad \cdots \quad v_n]$
- A linearly independent subset of $\{v_1, \cdots, v_n\}$ consists of the columns containing the leading 1's in $\text{ref}(A)$.

## 5. Spanning set

**Example 5.1.** *Find a spanning set for the plane*

$$x + 2y - 3z = 0$$

*in* $\mathbb{R}^3$*.*

**Solution.** This plane actually gives a homogeneous linear system

$$x + 2y - 3z = 0.$$

Solving it, we obtain the general expression for the solutions

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Then $\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \}$ is a spanning set of the plane $x + 2y - 3z = 0$. ◀

**Theorem 5.2.** *Given an* $m \times n$ *matrix* $A$*, or equivalently a homogeneous linear system* $Ax = 0$*, then*

$$\text{rank}(A) + \dim(\textit{Null space of } A) = n = \textit{the number of unknowns.}$$

**Summary:** How to find a spanning set of a subspace in $\mathbb{R}^n$?

- A subspace of $\mathbb{R}^n$ is usually given by a homogeneous linear system $Ax = 0$, where $A$ is a $m \times n$ matrix.
- Solving this linear system, the solutions can be expressed as

  $$\vec{x} = c_1 v_1 + \cdots c_k v_k,$$

  where $c_1, \cdots, c_k$ are free parameters, and $v_1, \cdots, v_k$ are fixed vectors in $\mathbb{R}^n$. Here

  $$k = n - \text{rank}(A)$$

  by Theorem 5.2.
- Then $\{v_1, \cdots, v_k\}$ is a spanning set of the subspace.

**Remark 5.3.** $\{v_1, \cdots, v_k\}$ *is indeed a basis of this subspace!!*

**Theorem 5.4.** *If a vector space* $V$ *is of dimension* $n$ *and* $\{v_1 \cdots, v_n\}$ *is a subset of* $V$*, then the following statements are equivalent.*

1. $\{v_1 \cdots, v_n\}$ *is linearly independent.*
2. $\{v_1 \cdots, v_n\}$ *is a spanning set.*
3. $\{v_1 \cdots, v_n\}$ *is a basis.*

**Example 5.5.** *Is*

$$\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}6\\0\\2\end{bmatrix}\}$$

*a spanning set for the plane*

$$x + 2y - 3z = 0$$

*in* $\mathbb{R}^3$.

**Solution.** First, it is an easy task to check all three vectors in $\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}6\\0\\2\end{bmatrix}\}$ satisfy $x + 2y - 3z = 0$. So they all belong to the plane. Let

$$A = \begin{bmatrix}1 & 3 & 6\\1 & 0 & 0\\1 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}.$$

Therefore, $\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}\}$ is linearly independent. Recall by Theorem 5.2, the dimension of the plane $x + 2y - 3z = 0$ is 2. By Theorem 5.4, $\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}\}$ is a spanning set. ◀

**Summary:** How to determine whether a given set $\{v_1, \cdots, v_k\}$ is a spanning set of a subspace $S$ of $\mathbb{R}^n$?

- Find the homogeneous linear system $Ax = 0$, where $A$ is an $m \times n$ matrix, representing the subspace $S$.
- Verify if $v_i$'s are solutions to $Ax = 0$. If one of $v_i$'s is not a solution, then this is not a spanning set.
- If all $v_i$'s are solutions, then let
$$A = \begin{bmatrix}v_1 & \cdots & v_k\end{bmatrix}.$$
- Pick up a linearly independent subset out of $\{v_1, \cdots, v_k\}$. Recall a linearly independent subset of $\{v_1, \cdots, v_k\}$ consists of the columns containing the leading 1's in ref$(A)$.
- Use Theorem 5.2 to find out the dimension of the subspace $S$.
- If dim$S$ =the number of vectors in the linearly independent subset of $\{v_1, \cdots, v_k\}$, then $\{v_1, \cdots, v_k\}$ is a spanning set. Otherwise, not.

**Summary:** How to determine whether a given set $\{v_1, \cdots, v_n\}$ is a spanning set of $\mathbb{R}^m$? (NOT a subspace)!

- Let

$$A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}.$$

- If $\text{rank}(A)$ = the number of rows of $A$, or equivalently, there is no bottom zero row in $\text{ref}(A)$, then $\{v_1, \cdots, v_n\}$ is a spanning set. Otherwise, not.

**Remark 5.6.** *When $m = n$, $A$ is a square matrix. In this case, the last step can be replaced by computing the determinant of $A$. More precisely, if $\det(A) \neq 0$, then $\{v_1, \cdots, v_n\}$ is spanning set. Otherwise, not.*

## 6. Dimensions and bases

**Proposition 6.1.** *$V$ is of dimension $n$. Then*

- *any linearly independent set cannot contain more than $n$ vectors;*
- *any spanning set must contain at least $n$ vectors;*
- *any basis contains exactly $n$ vectors.*

**Remark 6.2.** *A basis can be considered as a "maximal" linearly independent set, or a "minimal" spanning set.*

**Proposition 6.3.** *$S$ is a subspace of $V$. Then $\text{dim}S \leq \text{dim}V$. If $\text{dim}S = \text{dim}V$, then $S = V$.*

**Theorem 6.4.** *$S$ is a subspace of $V$. Then any basis of $S$ can be extended to a basis of $V$.*

**Question:** Given a basis of a subspace $S$, how to extend it to a basis of $V$?

**Example 6.5.** *Extend the basis of* $\text{span}(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix})$ *to a basis of* $\mathbb{R}^3$.

**Solution.** We first check

$$\begin{bmatrix} -2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}\}$ is a basis for span($\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}$). To find the third vector extending $\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}\}$ into a basis for $\mathbb{R}^3$, we look at

$$A = \begin{bmatrix}-2 & 3 & 1 & 0 & 0\\ 1 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 1\end{bmatrix}.$$

The last three column vectors in $A$ is the standard basis for $\mathbb{R}^3$. Thus, the five column vectors of $A$ is a spanning set of $\mathbb{R}^3$. Moreover,

$$A \sim \begin{bmatrix}1 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 2 & 3\end{bmatrix}.$$

From this observation, we know that the first three columns of $A$ are linearly independent, and thus$\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}\}, \begin{bmatrix}1\\0\\0\end{bmatrix}\}$ form a basis for $\mathbb{R}^3$. ◀

**Summary:** How to extend a basis for a subspace $S$ to a basis $\mathbb{R}^n$?

- Find a basis $\{v_1, \cdots, v_k\}$ for $S$.
- Let $A = \begin{bmatrix}v_1 & v_2 & \cdots & v_k & u_1 & u_2 & \cdots & u_n\end{bmatrix}$. Here $\{u_1, \cdots, u_n\}$ is a basis (usually, we take this set to be the standard basis) of $\mathbb{R}^n$.
- A basis of $V$ is the column vectors corresponding to the columns containing the leading 1's in ref$(A)$ (these columns will include $\{v_1, \cdots, v_k\}$).

**Summary:** How to find a basis for $\mathbb{R}^n$ or a subspace $S$ of $\mathbb{R}^n$?

- Find a spanning set $\{v_1, \cdots, v_n\}$.
- Let $A = \begin{bmatrix}v_1 & v_2 & \cdots & v_n\end{bmatrix}$.
- A basis set, or equivalently a linearly independent subset of $\{v_1, \cdots, v_n\}$, is the column vectors corresponding to the columns containing the leading 1's in ref$(A)$.

**Summary:** How to determine if a set $\{v_1, \cdots, v_m\}$ is a basis of $\mathbb{R}^n$?

- If $m \neq n$, this is not a basis.
- If $m = n$, let $A = \begin{bmatrix}v_1 & v_2 & \cdots & v_n\end{bmatrix}$.
- If $\det(A) \neq 0$, or equivalently, $\text{rank}(A) = n = m$, this is a basis. Otherwise, it is not.

## 7. Relationship between spanning sets, liner independence, and bases

Suppose $V$ is an $m$-dimensional vector space and $S = \{v_1, \cdots, v_n\}$ is a set of vectors in $V$. Then

- if $n > m$, then $S$ is linearly dependent;
- if $m > n$, then $S$ is not a spanning set;
- if $n \neq m$, then $S$ is not a basis.

$S = \{v_1, \cdots, v_n\}$ is a basis for $V$ means

- $S$ is a "maximal" linearly independent subset of $V$, i.e., for any $u \in V$, $\{v_1, \cdots, v_n, u\}$ becomes linearly dependent;
- $S$ is a "minimal" spanning set of $V$, i.e., after removing any vector $v_k$ from $S$, the set $\{v_1, \cdots, v_{k-1}, v_{k+1}, \cdots, v_n\}$ is not spanning set anymore.

## 8. Rank, Row and column spaces

$$A_{m\times n} = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix} = \begin{bmatrix} c_1 & \cdots c_n \end{bmatrix} \text{ is a } m \times n \text{ matrix.}$$

Note that

$$\text{colspace}(A) = \text{rowspace}(A^T), \quad \text{colspace}(A^T) = \text{rowspace}(A).$$

**Remark 8.1.**

- *colspace*$(A)$ *is a subspace of* $\mathbb{R}^m$*.*
- *rowspace*$(A)$ *is a subspace of* $\mathbb{R}^n$*.*

**Remark 8.2.** *rank*$(A)$ = *the dimension of colspace*$(A)$ = *the dimension of rowspace*$(A)$

**Summary:** How to find a basis for colspace$(A)$?

- The basis for colspace$(A)$ consists of the columns containing the leading 1's in ref$(A)$.

**Summary:** How to find a basis for rowspace$(A)$?

- The nonzero rows in ref$(A)$ form a basis for rowspace$(A)$. (Note that these rows are not from the original rows of $A$.) Or

- The columns containing the leading 1's in ref($A$) forms a basis for colspace($A^T$). Taking transpose of these columns, we obtain a basis for rowspace($A$). (These rows are not from the original rows of $A$.)