A PROOF FOR THE CRAMER’S RULE

Proof. Suppose that $A$ is an $n \times n$ invertible matrix. We look at the linear system $AX = b$. Then this system has a unique solution

$$X = A^{-1}b = \frac{1}{|A|} \text{adj} A \cdot b = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} \cdot b.$$ 

Here $\cdot$ denotes matrix multiplication. By the formula for matrix multiplication, the $k$-th unknown, $x_k$, can be written as

$$x_k = \frac{1}{|A|} \sum_{i=1}^{n} A_{ik}b_i = \frac{1}{|A|} \det \begin{bmatrix} c_1 & \cdots & c_{k-1} & b & c_{k+1} & \cdots & c_n \end{bmatrix}.$$ 

Here $c_i$ denotes the $i$-th column of $A$. The second equality follows from the cofactor expansion of the matrix $\begin{bmatrix} c_1 & \cdots & c_{k-1} & b & c_{k+1} & \cdots & c_n \end{bmatrix}$ along the $k$-th column. \qed