CHAPTER 5 REVIEW

Throughout this note, we assume that V and W are two vector spaces with $\dim V = n$ and $\dim W = m$. $T : V \to W$ is a linear transformation.

1. A map $T : V \to W$ is a linear transformation if and only if

$$T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2),$$

for all $v_1, v_2 \in V$ and all scalars c_1, c_2.

Every linear transform $T : \mathbb{R}^n \to \mathbb{R}^m$ can be expressed as the matrix product with an $m \times n$ matrix:

$$T(v) = [T]_{m\times n}v = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} v,$$

for all n-column vector v in $\mathbb{R}^n$. Then matrix $[T]_{m\times n}$ is called the matrix of transformation T, or the matrix representation for T with respect to the standard basis.

Remark 0.1. *More generally, given arbitrary basis $\mathcal{B} = \{v_1, \cdots, v_n\}$ of $\mathbb{R}^n$,*

$$T(v) = [T_{\mathcal{B}}]_{m\times n}v = \begin{bmatrix} T(v_1) & T(v_2) & \cdots & T(v_n) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

where $v = c_1v_1 + \cdots + c_nv_n$. $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ can be considered as the coordinates of v with respect to the basis $\{v_1, \cdots, v_n\}$.

Example 0.2. *Let $T : \mathbb{P}_3 \to \mathbb{P}_2$ be a linear transformation defined by*

$$T(a_0 + a_1t + a_2t^2 + a_3t^3) = (a_0 + a_3) + (a_1 + a_2)t + (a_0 + a_1 + a_2 + a_3)t^2.$$

Then the matrix of T relative to the bases. If we choose the standard bases

$$\{t^3, t^2, t, 1\}, \quad \{t^2, t, 1\}$$

for $\mathbb{P}_3$ and $\mathbb{P}_2$, respectively. Then

$$[T] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

2. How to find the image of a vector under a linear transformation.

Example 0.3. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by*

$$T(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}-3\\-3\end{bmatrix}, \quad T(\begin{bmatrix}2\\1\end{bmatrix}) = \begin{bmatrix}4\\2\end{bmatrix}.$$

Find $T(\begin{bmatrix}4\\3\end{bmatrix})$.

Solution. We first try to find constants c_1, c_2 such that

$$\begin{bmatrix}4\\3\end{bmatrix} = c_1 \begin{bmatrix}1\\1\end{bmatrix} + c_2 \begin{bmatrix}2\\1\end{bmatrix}.$$

It is not a hard job to find out that

$$c_1 = 2, \quad c_2 = 1.$$

Therefore,

$$T(\begin{bmatrix}4\\3\end{bmatrix}) = \begin{bmatrix}-3 & 4\\-3 & 2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-2\\-4\end{bmatrix}.$$

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Example 0.4. *T is a linear transformation from $\mathbb{P}_2$ to $\mathbb{P}_2$, and*

$$T(x^2-1) = x^2+x-3, \quad T(2x) = 4x, \quad T(3x+2) = 2x+6.$$

Find $T(1)$, $T(x)$, and $T(x^2)$.

Solution. We identify T as a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ by the map

$$ax^2+bx+c \mapsto \begin{bmatrix}a\\b\\c\end{bmatrix}.$$

By the given conditions, we have

$$T(1,0,-1) = (1,1,-3), \quad T(0,2,0) = (0,4,0), \quad T(0,3,2) = (0,2,6).$$

We immediately have

$$T(0,1,0) = \frac{1}{2}T(0,2,0) = (0,2,0).$$

Let

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 2 & 3\\-1 & 0 & 2\end{bmatrix}.$$

Solving

$$A\vec{x} = \begin{bmatrix}0\\0\\1\end{bmatrix},$$

we get $\vec{x} = \begin{bmatrix} 0 \\ -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$. Therefore,

$$T(0,0,1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 2 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}.$$

Finally,

$$T(1,0,0) = T(1,0,-1) + T(0,0,1) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Now we restore this result back to the space $\mathbb{P}_2$ and obtain

$$T(1) = -2x + 3, \quad T(x) = 2x, \quad T(x^2) = x^2 - x.$$

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Summary: Suppose the images of a basis $\mathcal{B} = \{v_1, \cdots, v_n\}$ are given, i.e., we known $T(v_1), \cdots, T(v_n)$. For a given vector v,

1. we identify T as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$;
2. write down the representation matrix $[T_{\mathcal{B}}]$;
3. find the coordinates of v, i.e., $v = c_1 v_1 + \cdots + c_n v_n$;
4. $T(v) = [T_{\mathcal{B}}] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$;
5. restore the result in $\mathbb{R}^n$ and $\mathbb{R}^m$ to the original vector spaces V and W.

If you are asked to find the images for all vectors in V,

1. we identify T as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$;
2. follow the steps given above to find $T(e_1), \cdots T(e_n)$;
3. write down the representation matrix $[T]$;
4. $T(v) = [T]v$;
5. restore the result in $\mathbb{R}^n$ and $\mathbb{R}^m$ to the original vector spaces V and W.

3. Kernel and Range

Example 0.5. *T is a linear transformation from $\mathbb{P}_1$ to $\mathbb{P}_2$. Moreover,*

$$T(a+bx) = (2a-3b) + (b-5a)x + (a+b)x^2.$$

Find Ker(T) *and* Rng(T)*.*

Solution. We identify T as a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. By the given conditions, we have

$$T(\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\-3\end{bmatrix}, \quad T(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}1\\-5\\2\end{bmatrix}.$$

So the representation matrix $[T]$ of T is

$$\begin{bmatrix}1 & 1\\1 & -5\\-3 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}.$$

Ker$(T) =$ Null space of $[T]$. For any $m \times n$ matrix A,

$$\text{rank}(A) + \dim(\text{Null space of } A) = n.$$

So Ker$(T) = \{0\}$.

On the other hand,

$$\begin{aligned}\text{Rng}(T) &= \{T(ax+b) : a, b \in \mathbb{R}\}\\ &= \{(2a-3b) + (b-5a)x + (a+b)x^2 : a, b \in \mathbb{R}\}\\ &= \{a(2-5x+x^2) + b(-3+x+x^2) : a, b \in \mathbb{R}\}\\ &= \text{span}\{2-5x+x^2, -3+x+x^2\}.\end{aligned}$$

dim(Rng(T)) = 2. ◀

Summary: Kernel

1. we identify T as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$;
2. find the representation matrix $[T] = [T(e_1) \quad \cdots \quad T(e_n)]$;
4. Ker(T) is the solution space to $[T]x = 0$.
5. restore the result in $\mathbb{R}^n$ to the original vector space V.

Example 0.6. *Find the range of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ whose standard representation matrix is given by*

$$A = [T] = \begin{bmatrix}1 & 1 & 2 & 3\\3 & 4 & -1 & 2\\-1 & -2 & 5 & 4\end{bmatrix}.$$

Solution. $\text{Rng}(T) = \text{colspace}([T])$. Note that the $\text{colspace}([T])$ consists of all the vectors $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that the homogeneous linear system $[T]x = v$ is consistent.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & \vdots & x \\ 3 & 4 & -1 & 2 & \vdots & y \\ -1 & -2 & 5 & 4 & \vdots & z \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 10 & \vdots & 4x - y \\ 0 & 1 & -7 & -7 & \vdots & y - 3x \\ 0 & 0 & 0 & 0 & \vdots & y + z - 2x \end{bmatrix}.$$

The above system is consistent if and only if $y + z - 2x = 0$, that is, $\text{colspace}([T])$ consists of all vectors $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying $y + z - 2x = 0$.

In other words, $\text{colspace}([T])$ is the plain $y + z - 2x = 0$. ◀

Remark 0.7. *This example actually gives an algorithm to find* $\text{colspace}(A)$ *and* $\text{rowspace}(A)$ *of a matrix* A.

Summary: Range

1. we identify T as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$;
2. find the representation matrix $[T] = \begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix}$;
3. $\text{Rng}(T) = \text{colspace}([T])$, which is a subspace of $\mathbb{R}^m$;
4. restore the result in $\mathbb{R}^m$ to the original vector space W.

4. How to find eigenvalues and eigenvectors/eigenspaces?

Example 0.8. *Find the eigenvalues and eigenspaces of the matrix*

$$A = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix}.$$

Determine A *is defective or not.*

Solution. The characteristic polynomial is given by

$$p(\lambda) = \det(A - \lambda I_3) = -(\lambda - 2)^2(\lambda + 1).$$

So the eigenvalues are $\lambda_1 = 2, \lambda_2 = -1$. Their multiplicities are $m_1 = 2, m_2 = 1$. Since

$$A - 2I_3 \sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace with respect to $\lambda_1 = 2$ is

$$E_1 = \text{span}\{\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\}.$$

Similarly, the eigenspace with respect to $\lambda_2 = -1$ is

$$E_2 = \text{span}\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\}.$$

We have $\dim E_i = m_i$ for $i = 1, 2$. So A is non-defective. ◀

Example 0.9. *Find the eigenvalues and eigenspaces of the matrix*

$$A = \begin{bmatrix} 6 & 5 \\ -5 & -4 \end{bmatrix}.$$

Determine A is defective or not.

Solution. The characteristic polynomial is given by

$$p(\lambda) = \det(A - \lambda I_2) = (\lambda - 1)^2.$$

So the eigenvalues are $\lambda_1 = 1$ with multiplicity $m_1 = 2$. Since

$$A - I_2 \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus, the eigenspace with respect to $\lambda_1 = 1$ is

$$E_1 = \text{span}\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}.$$

We have $\dim E_1 = 1 < m_1 = 2$. So A is defective. ◀

Summary:

1. Factorize the characteristic polynomial
$$p(\lambda) = \det(A - \lambda I) = \pm(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k}.$$
Then $\lambda_1, \cdots, \lambda_2$ are all the eigenvalues of A.
2. The eigenspace E_i with respect to the eigenvalue λ_i is the solution space to the homogeneous linear system $(A - \lambda_i I)x = 0$. All the non-zero vectors in E_i are the eigenvectors with respect to the eigenvalue λ_i.
3. A is non-defective if and only if $\dim E_i = m_i$ for $k = 1, \cdots, k$.

Remark 0.10. *An $n \times n$ matrix A is non-defective if A has n distinct roots. If A has some repeated root(s), then while check defectiveness A, one only need to check the dimension(s) of the eigenspace(s) E_i with respect to the eigenvalue(s) λ_i with multiplicity $m_i \geq 2$.*