Student Name ______________________________

Student ID ______________________________

Instructor ______________________________

Section number (3-digits, as in the table below) ____________

Mark Test 01 and Section Number on your scantron !

There are 25 questions on this exam. Each question is worth 8 points.

Exam Rules

1. Students may not open the exam until instructed to do so.
2. Students must obey the orders and requests by all proctors, TAs, and lecturers.
3. No student may leave in the first 20 min or in the last 10 min of the exam.
4. Books, notes, calculators, or any electronic devices are not allowed on the exam, and they should not even be in sight in the exam room. Students may not look at anybody else's test, and may not communicate with anybody else except, if they have a question, with their TA or lecturer.
5. After time is called, the students have to put down all writing instruments and remain in their seats, while the TAs will collect the scantrons and the exams.
6. Any violation of these rules and any act of academic dishonesty may result in severe penalties. Additionally, all violators will be reported to the Office of the Dean of Students.

I have read and understand the exam rules stated above:

STUDENT SIGNATURE: ______________________________

Section Numbers:

Bauman, Patricia: **071** Dadarlat, Marius: **033**
Gabrielov, Andrei: **031** Ulrich, Bernd: **121**
Zhilan Julie Feng: **061** Noparstak, Jakob: **173**
Ma, Linquan: **021** (10:30am) and **072** (9:30am)
Li, Dan **022** (02:30pm) and **052** (01:30pm)
Kelleher, Daniel: **131** (04:30pm) and **141** (03:30pm)
Gnang, Edinah **032** (09:30am) and **051** (10:30am)
Xu, Xiang **132** (MWF 03:30pm) and **162** (MWF 04:30pm)
Zhang, Xu: **171** (TR 01:30pm) and **172** (TR 12:00pm)

(1) If we solve the equation:

$$\begin{bmatrix} a+b & c-a \\ c+2a & 3 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 11 & 3 \end{bmatrix}$$

for a, b and c, the values of a and b are:

$a + b = 3$

$-a + c = -4$

$2a \quad c = 11$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ -1 & 0 & 1 & -4 \\ 2 & 0 & 1 & 11 \end{array}\right] \Rightarrow \begin{array}{l} a=5 \\ b=-2 \\ c=1 \end{array}$$

A. $a = 2,\ b = 1$

B. $a = 7,\ b = -4$

C. $a = 5,\ b = -2$

D. $a = -1,\ b = 4$

E. There is no solution

(2) Let $A = \begin{bmatrix} 1 & -1 & 5 \\ -2 & 3 & 1 \\ 4 & 1 & 2 \\ 1 & -3 & 8 \end{bmatrix}$. Compute the (2,3) entry of $A^T A$.

A. 7

B. 14

C. -3

D. -14

E. -24

(3) Which of the following sets of vectors span $\mathbf{R}^3$?

A. $\begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

B. Too few vectors

C $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 3 & 0 \end{vmatrix} = 0$

D $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 4 & 2 & 0 \end{vmatrix} = 0.$

A. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\}$

B. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$

C. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

D. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

E. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

(4) Which of the following is a basis for the subspace of $\mathbf{R}^3$ spanned by

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\} ?$$

$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & -4 & -2 & 0 \\ 0 & -2 & -1 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The 1st, 2nd, 4th columns are α I.

A. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$

B. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}$

C. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}$

D. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$

E. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$

(5) If $L : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

then $a + b + c$ is equal to:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 2 \cdot \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

A. 8
B. 10
C. 7
D. 4
E. 5

(6) If $y = ax + b$ is the least square fit line for the points $(-1, 3)$, $(0, 2)$, $(1, 4)$, find $a + b$.

We need to find the least square solution to

$$AX = b \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \quad X = \begin{bmatrix} b \\ a \end{bmatrix}$$

$$b = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

$$A^T A X = A^T b \iff \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$$

$$\Rightarrow b = 3 \qquad a = \frac{1}{2}$$

A. 2
B. 7/2
C. 3
D. 5
E. 9/2

(7) Consider the matrix

$$A = \begin{bmatrix} t & 1 & t & t \\ 0 & 1 & 1 & t \\ 0 & 0 & -1 & t \\ 0 & 0 & t & 1 \end{bmatrix},$$

where t is a real number. Then A is nonsingular if and only if

A. $t \neq 0$
B. $t \neq 0$ and $t \neq 1$
C. $t \neq 1$ and $t \neq -1$
D. $t \neq 0, t \neq 1$, and $t \neq -1$
E. t is any real number

$|A| = t \begin{vmatrix} -1 & t \\ t & 1 \end{vmatrix} = t(-1-t^2)$

$|A| \neq 0 \Rightarrow t \neq 0$

(8) For a nonsingular 6×6 matrix A, the determinant of the adjoint matrix adj(A) is

A. $\det(A)$
B. $\det(A^5)$
C. $\det(A^4)$
D. $\det(A^2)$
E. $\det(A^{-1})$

$\text{adj}\, A = |A| A^{-1}$

$|\text{adj}\, A| = \det(|A| A^{-1})$

$= |A|^6 |A^{-1}| = |A|^6 \frac{1}{|A|} = |A|^5 = |A^5|$

(9) Let V be the set of all strictly positive numbers in $\mathbf{R}$, and let $\oplus$ and $\odot$ be defined by

$$\mathbf{a} \oplus \mathbf{b} = ab, \quad \text{for any } \mathbf{a}, \mathbf{b} \text{ in } V \text{ (that is, } a,\ b \text{ are strictly positive numbers),}$$

$$c \odot \mathbf{a} = a^c, \quad \text{for any } \mathbf{a} \text{ in } V \text{ and } c \text{ in } \mathbf{R}.$$

Which of the following statements are true?

(i) For any $\mathbf{a}, \mathbf{b}$ in V and any c in $\mathbf{R}$, $c \odot (\mathbf{a} \oplus \mathbf{b})$ belongs to V.

(ii) Under the given operations, the $\mathbf{0}$ element is the real number 1.

(iii) There is at least one element $\mathbf{a}$ in V for which there is no element $-\mathbf{a}$ such that $(-\mathbf{a}) \oplus \mathbf{a} = \mathbf{0}$.

A. (i) only

B. (ii) only

C. (i) and (ii) only

D. (i) and (iii) only

E. All of them

(10) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Which of the following statements are true?

(i) A basis for span S is $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(ii) The vector $\mathbf{u} = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix}$ belongs to span S.

(iii) S is a linearly dependent set.

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 2 & -2 & -4 \\ -3 & -2 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -2 \\ -3 & -2 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \text{(i) \& (ii)}$$

A. (i) only

B. (iii) only

C. (i) and (iii) only

D. (ii) and (iii) only

E. All of them

(11) Let W be the vector space spanned by the vectors:

$$\mathbf{u}_1 = \begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix}1 & 1 & 0 & 1\end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}.$$

Apply the Gram-Schmidt process to the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ (in this order) to find an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ of W. What is $\mathbf{w}_3$?

$V_1 = U_1$

$V_2 = U_2 - \frac{(U_2, V_1)}{\|V_1\|^2} V_1 = U_2.$

$V_3 = U_3 - \frac{(U_3, V_1)}{\|V_1\|^2} V_1 - \frac{(U_3, V_2)}{\|V_2\|^2} V_2$

$= \begin{bmatrix}1\\1\\0\\1\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$

$= \begin{bmatrix}0\\1/2\\-1/2\\0\end{bmatrix}$ normalize $V_3 \Rightarrow W_3 = \frac{V_3}{\|V_3\|} = \begin{bmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}$

A. $\begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}$

B. $\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}$

C. $\begin{bmatrix}0 & 1/\sqrt{2} & 1/\sqrt{2} & 0\end{bmatrix}$

D. $\begin{bmatrix}0 & 1/\sqrt{2} & -1/\sqrt{2} & 0\end{bmatrix}$

E. $\begin{bmatrix}1-\sqrt{2} & -1/\sqrt{2} & -\sqrt{2} & 1-\sqrt{2}\end{bmatrix}$

(12) Consider the subspace W of $\mathbf{R}_4$:

$$W = \text{span}\left\{\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 3 & 1\end{bmatrix}, \quad \begin{bmatrix}-1 & -1 & 2 & 0\end{bmatrix}, \quad \begin{bmatrix}1 & 1 & 4 & 2\end{bmatrix}\right\}.$$

What is the dimension of $W^{\perp}$?

rank $\begin{bmatrix}1 & 1 & 1 & 1\\0 & 0 & 3 & 1\\-1 & -1 & 2 & 0\\1 & 1 & 4 & 2\end{bmatrix} = 2.$

A. 0

B. 1

C. 2

D. 3

E. 4

$\Downarrow$

$\dim W^{\perp} = 4 - 2 = 2.$

(13) Consider the homogeneous linear system

$$\begin{aligned} a + 2b + 3c + 4d + 5e &= 0 \\ a \qquad + 2c \qquad + 3e &= 0 \\ b + 2c + 3d \qquad &= 0 \end{aligned}$$

Then the dimension of the solution space is

A. 0
B. 1
C. 2
D. 3
E. 4

(14) Suppose A is a 3×5 matrix such that $rank(A) = 3$. Which of the following is TRUE?

A. The rank of A^T is 5
B. The nullity of A^T is 2
C. $A\mathbf{x} = \mathbf{0}$ only has trivial solution
D. The rows of A are linearly dependent
E. The columns of A are linearly dependent

rank $A = 3$ $\Rightarrow$ rank $A^T = 3$

$\Downarrow$

$3 =$ dim (rowspace (A)) $=$ dim (colspace (A))

$\Downarrow$

nullity of $A = 2$

So rows of A are L.I.

the columns of A are linearly dep.

(15) The eigenvectors of $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with eigenvalues 3 and -1 respectively. If $x_1(t)$, $x_2(t)$ satisfy $x_1(0) = 1$, $x_2(0) = 3$ and

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

compute $x_1(1) + x_2(1)$.

$$X(t) = b_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + b_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}.$$

$$X(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\Rightarrow b_1 = 2 \qquad b_2 = 1$$

$$X(1) = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-1}$$

$$X_1(1) + X_2(1) = (2e^3 - e^{-1}) + (2e^3 + e^{-1}) = 4e^3$$

A. $4e^3$
B. $4e^3 - 2e^{-1}$
C. $e^3 - 2e^{-1}$
D. $e^3 - e^{-1}$
E. $e^3 - 4e^{-1}$

(16) The dimension of the vector space of all 4×4 symmetric matrices with real entries is equal to:

A. 6
B. 8
C. 9
D. 10
E. 12

Recall any 4×4 sym matrix is of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$$

So we only need to determine the entries above (and including) the main diagonal. There are 10 entries.

(21) For which values of a does Gaussian elimination applied to

$$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix}$$

fail to give three pivots (leading 1's)?

A has 3 pivots $\Leftrightarrow$ $|A| \neq 0$

$$0 = |A| = \begin{vmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & a-2 & a-3 \end{vmatrix} = a\left[(a-2)(a-3) - (a-2)\right]$$

$$= a(a-2)(a-4)$$

$a = 0, 2, 4$

A. $1, 2, 3$
B. $0, 1, 3$
C. $0, 2, 3$
D. $0, 3, 4$
E. $0, 2, 4$

(22) If

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{has inverse} \quad A^{-1} = \begin{bmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix},$$

then $b + c$ must be equal to:

Use adj A

$$|A| = \begin{vmatrix} 0 & 3 & -2 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} = 4$$

The (1, 2) & (1, 3) entry of adj A are

$$A_{21} = -\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = 2 \qquad A_{31} = \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} = 1$$

$$b + c = \frac{A_{21} + A_{31}}{4} = \frac{3}{4}$$

A. $3/4$
B. $5/4$
C. $1/4$
D. $-3/4$
E. $7/4$

(23) Let $A = \begin{bmatrix} 1+i & 1 \\ 1 & 1-i \end{bmatrix}$. Then A^{-1} is given by

$|A| = 1$

$A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \begin{vmatrix} 1-i & -1 \\ -1 & 1+i \end{vmatrix}$

A. $\begin{bmatrix} 1+i & 1 \\ 1 & 1-i \end{bmatrix}$

B. $\begin{bmatrix} 1-i & -1 \\ -1 & 1+i \end{bmatrix}$

C. $\begin{bmatrix} (1+i)/2 & 1/2 \\ 1/2 & (1-i)/2 \end{bmatrix}$

D. $\begin{bmatrix} 1-i & 1 \\ 1 & 1+i \end{bmatrix}$

E. A^{-1} does not exist.

(24) Let $\mathbf{u}$ and $\mathbf{v}$ be orthogonal vectors in $\mathbf{R}^5$ such that $||\mathbf{u}|| = \sqrt{7}$, $||\mathbf{v}|| = 3$. Then $||2\mathbf{u} - \mathbf{v}||$ equals

$(2u - v, 2u - v) = 4\|u\|^2 + \|v\|^2 - 4\underbrace{(2u, v)}_{=0}$

$= 28 + 9 = 37$

A. $\sqrt{19}$

B. $\sqrt{23}$

C. 5

D. $\sqrt{37}$

E. 6

(25) Let A be a 7×3 matrix such that its null space is spanned by the vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

The rank of A is:

A. 1
B. 2
C. 3
D. 4
E. 6

$$\text{rank} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = 2 \Rightarrow \text{nullity of } A = 2$$

$$\Rightarrow \text{rank } A = 3 - 2 = 1$$