EXAMPLES OF SECTIONS 1.9

Question 1. Solve $\frac{y}{x} + 6x + (\ln|x| - 2)y' = 0$.

Question 2. Find b such that $(xy^2 + bx^2y)dx + (x+y)x^2dy$ is exact and solve the equation.

Question 3. Find the integrating fact for $(3x^3 + y)dx + (2x^2y - x)dy = 0$ and solve the equation.

Solutions.

1. First, one checks

$$\frac{\partial}{\partial y}M = \frac{1}{x} = \frac{\partial}{\partial x}N,$$

and thus this equation is exact. Set

$$\phi(x,y) = \int M(x,y)\,dx + h(y) = y\ln|x| + 3x^2 + h(y),$$

and differentiate ϕ w.r.t. y

$$\frac{\partial}{\partial y}\phi(x,y) = \ln|x| + +h'(y) = N(x,y) = \ln|x| - 2.$$

We thus obtain $h'(y) = -2$, which yields $h(y) = -2y$. In summary,

$$\phi(x,y) = y\ln|x| + 3x^2 - 2y = C,$$

or equivalently

$$y(x) = \frac{C - 3x^2}{\ln|x| - 2}$$

is the general solution.

2. We need

$$\frac{\partial}{\partial y}M = 2xy + bx^2 = \frac{\partial}{\partial x}N = 3x^2 + 2xy,$$

which clearly implies $b = 3$. As above,

$$\begin{aligned}
\phi(x,y) = \int M(x,y)\,dx + h(y) &= \frac{1}{2}x^2y^2 + x^3y + h(y) \\
&\Rightarrow \frac{\partial}{\partial y}\phi(x,y) = x^2y + x^3 + h'(y) = x^3 + x^2y \\
&\Rightarrow h'(y) = 0 \Rightarrow h(y) = 0 \\
&\Rightarrow \phi(x,y) = \frac{1}{2}x^2y^2 + x^3y = C.
\end{aligned}$$

3. $(3x^3+y)dx+(2x^2y-x)dy=0$ Let $M(x,y)=3x^3+y$ and $N(x,y)=2x^2y-x$. Then

$$\partial_y M-\partial_x N=2(1-2xy)\Longrightarrow \frac{\partial_y M-\partial_x N}{N}=-\frac{2}{x}.$$

Then

$$I(x)=e^{\int \frac{2}{x}\,dx}=\frac{1}{x^2}.$$

The equation

$$(3x-\frac{y}{x^2})dx+(2y-\frac{1}{x})dy=0$$

is exact. Following the same procedure as above, we get

$$\phi(x,y)=\frac{3}{2}x^2+y^2-\frac{y}{x}=C$$

is the general solution.