EXAMPLES OF SECTIONS 4.2

Question 1. Consider the set V of all triples (x, y, z) such that $x = 3$. Is V a vector space?

Question 2. Consider the set V of solutions to the n-th order linear differential equation

$$a_0(x)y^{(n)}(x) + \cdots a_{n-1}(x)y(x) + a_n(x) = 0$$

on some interval I. Is V a vector space under function addition and scalar multiplication?

SOLUTIONS.

1. First notice that elements of V can be written as $(3, y, z)$. In order for V to be a vector space, there must exist a zero element, i.e., an element $q = (q_1, q_2, q_3)$ such that $q \in V$ and $q + u = u$ for every $u \in V$. But if $q \in V$ then it can be written as $q = (3, q_2, q_3)$, and it follows that

$$q + u = (3, q_2, q_3) + (3, u_2, u_3) = (6, q_2 + u_2, q_3 + u_3) \neq (3, u_2, u_3).$$

Therefore V it is not a vector space.

2. It is very hard to verify that V satisfies criteria A3-A4 and A7-A10. The zero element in V is the constant zero solution $y \equiv 0$, and the additive inverse of a solution $y(x)$ is simply $(-y)(x) = -y(x)$. It only remains to show that V is closed under function addition and scalar multiplication.

Suppose that y_1 and y_2 are two solutions to

$$a_0(x)y^{(n)}(x) + \cdots a_{n-1}(x)y(x) + a_n(x) = 0$$

on I. Then we have

$$a_0(x)y_1^{(n)}(x) + \cdots a_{n-1}(x)y_1(x) + a_n(x) = 0$$

and

$$a_0(x)y_2^{(n)}(x) + \cdots a_{n-1}(x)y_2(x) + a_n(x) = 0.$$

Adding these two equations together yields

$$a_0(x)(y_1 + y_2)^{(n)}(x) + \cdots a_{n-1}(x)(y_1 + y_2)(x) + a_n(x) = 0.$$

Similarly, for any $c \in \mathbb{R}$

$$a_0(x)cy^{(n)}(x) + \cdots a_{n-1}(x)cy(x) + a_n(x) = 0$$

if y solves the linear differential equations. So the closedness of function addition and scalar multiplication is assured.