10.8. Laplace Equation

Potential Equation

- The potential function of a particle in free space acted on only by gravitational forces satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0$$

  and hence Laplace's equation is often referred to as the **potential equation**.
- In elasticity, the displacements that occur when a perfectly elastic bar is twisted are described in terms of the so-called warping function, which also satisfies

$$u_{xx} + u_{yy} = 0$$

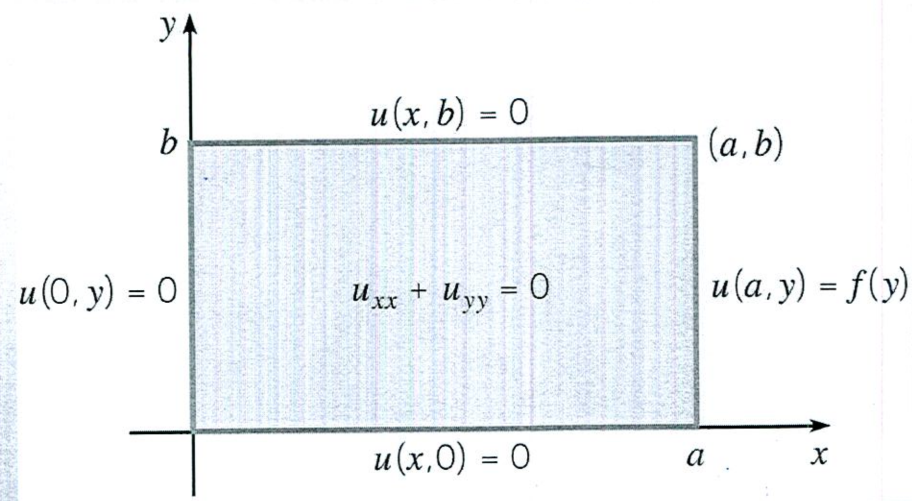
- There are many applications of Laplace's equation; see text.
- We will focus on the two-dimensional equation.

Dirichlet Problem for a Rectangle

- Consider the following Dirichlet problem on a rectangle:

$$u_{xx} + u_{yy} = 0, \quad 0 < x < a, \qquad 0 < y < b$$
$$u(x,0) = 0, \quad u(x,b)) = 0, \qquad 0 < x < a$$
$$u(0,y) = 0, \quad u(a,y) = f(y), \quad 0 \le y \le b$$

where f is a given function on $0 \le y \le b$.



(i) set function value at the boundary :
⇒ Dirichlet prob.

(ii) set derivative value at the boundary
⇒ Neumann prob.

(iii) $u(0) = 0$, $u_x(b) = c$
⇒ Mixed problem.

Separation of Variables Method

$u(x,0) = X(x)Y(0) = 0 \Rightarrow Y(0)=0$

$u(x,b) = \cdots \Rightarrow Y(b)=0$

$u(0,y) = X(0)Y(y) = 0 \Rightarrow X(0)=0$

$u(a,y) = X(a)Y(y) = f(y)$

$\Rightarrow$ Not possible "for the moment".

- We begin by assuming

$$u(x,y) = X(x)Y(y)$$

- Substituting this into our differential equation

$$u_{xx} + u_{yy} = 0$$

we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

or

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda \Rightarrow \begin{array}{l} X'' - \lambda X = 0, \quad X(0) = 0 \\ Y'' + \lambda Y = 0, \quad Y(0) = 0,\ Y(b) = 0 \end{array}$$

where λ is a constant.

- We next consider the boundary conditions.

(i) The sol'n to $Y'' + \lambda Y = 0$ with $Y(0) = 0$, $Y(b) = 0$

is: $\lambda_n = \left(\frac{n\pi}{b}\right)^2$, $Y_n(y) = \sin\left(\frac{n\pi y}{b}\right)$

(ii) $X'' - \left(\frac{n\pi}{b}\right)^2 X = 0$, $X(0) = 0$

char. eqn: $r^2 - \left(\frac{n\pi}{b}\right)^2 = 0 \Rightarrow r = \pm \frac{n\pi}{b}$

$$X(x) = \tilde{C}_1 e^{\frac{n\pi}{b}x} + \tilde{C}_2 e^{-\frac{n\pi}{b}x}$$

$$= C_1 \sinh \frac{n\pi}{b}x + C_2 \cosh \frac{n\pi}{b}x$$

$$\left(\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \right)$$

$$0 = X(0) = \tilde{C}_1 \cdot 1 + \tilde{C}_2 \cdot 1$$
$$= C_1 \cdot 0 + C_2 \cdot 1 \qquad \Rightarrow C_2 = 0$$

$$\Rightarrow X_n(x) = C_1 \sinh \frac{n\pi}{b} x.$$

fund. sol'ns are:

$$\sinh \frac{n\pi}{b}x \cdot \sin \frac{n\pi y}{b}$$

Initial Condition (6 of 8)

- Thus

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi x/b)\sin(n\pi y/b)$$

where the c_n are chosen so that the initial condition is satisfied:

$$u(a,y) = f(y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi a/b)\sin(n\pi y/b)$$

- Hence

$$c_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y)\sin(n\pi y/b)\,dy$$

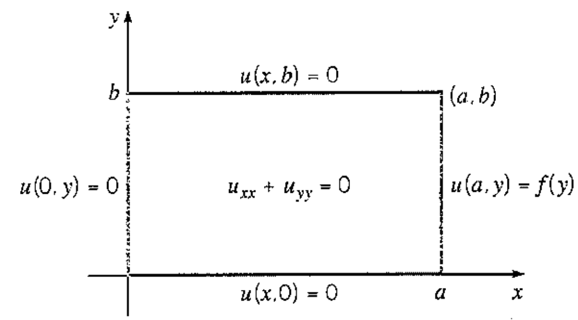
or

$$c_n = \frac{2}{b}\sinh\left(\frac{n\pi a}{b}\right)^{-1}\int_0^b f(y)\sin(n\pi y/b)\,dy$$

Solution (7 of 8)

- Therefore the solution to the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad 0 < x < a, \qquad 0 < y < b$$
$$u(x,0) = 0, \quad u(x,b) = 0, \qquad 0 < x < a$$
$$u(0,y) = 0, \quad u(a,y) = f(y), \quad 0 \le y \le b$$



is given by

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi x/b)\sin(n\pi y/b)$$

where

$$c_n = \frac{2}{b}\sinh\left(\frac{n\pi a}{b}\right)^{-1} \int_0^b f(y)\sin(n\pi y/b)\,dy$$

Ex 1. $f(y) = 5 \sin \frac{3\pi y}{b} + 7 \sin \frac{4\pi y}{b}$

$u(a,y) = f(y)$

$\Downarrow$

$$\sum_{n=1}^{\infty} C_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b} = 5 \sin \frac{3\pi y}{b} + 7 \sin \frac{4\pi y}{b}$$

$\Downarrow$

$$\begin{cases} C_3 \sinh \frac{3\pi a}{b} = 5 \\ C_4 \sinh \frac{4\pi a}{b} = 7 \\ C_n = 0 \qquad n \neq 3, 4 \end{cases}$$

$$\Rightarrow u(x,y) = \frac{5}{\sinh \frac{3\pi a}{b}} \sinh \frac{3\pi x}{a} \sin \frac{3\pi y}{b}$$

$$+ \frac{7}{\sinh \frac{4\pi a}{b}} \sinh \frac{4\pi x}{a} \sin \frac{4\pi y}{b}.$$

Example 1: Dirichlet Problem (1 of 2)

- Consider the vibrating string problem of the form

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 3, \qquad 0 < y < 2$$

$$u(x,0) = 0, \;\; u(x,2) = 0, \qquad 0 < x < 3$$

$$u(0,y) = 0, \;\; u(3,y) = f(y), \;\; 0 \le y \le 2$$

where

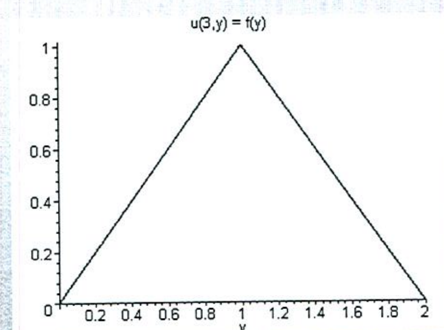
$$f(y) = \begin{cases} y, & 0 \le y \le 1 \\ 2 - y, & 1 \le y \le 2 \end{cases}$$

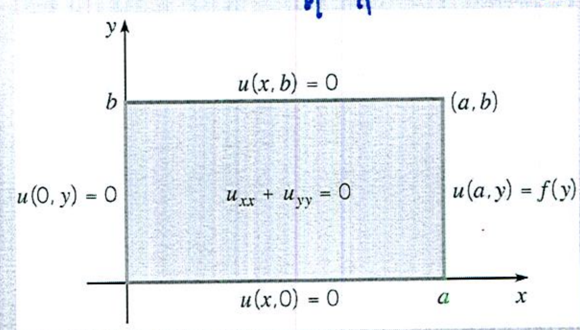
$$\int_0^2 f(y) \sin \frac{n\pi y}{2}\, dy$$

$$= \int_0^1 y \sin \frac{n\pi y}{2}\, dy + \int_1^2 (2-y) \sin \frac{n\pi y}{2}\, dy$$

$$= \cdots$$

$$= \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2}$$



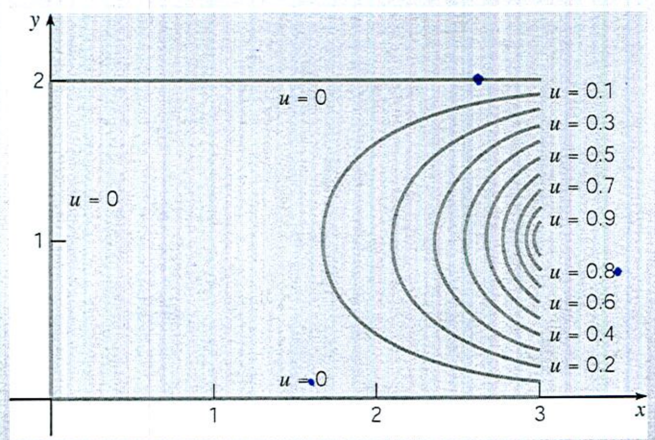


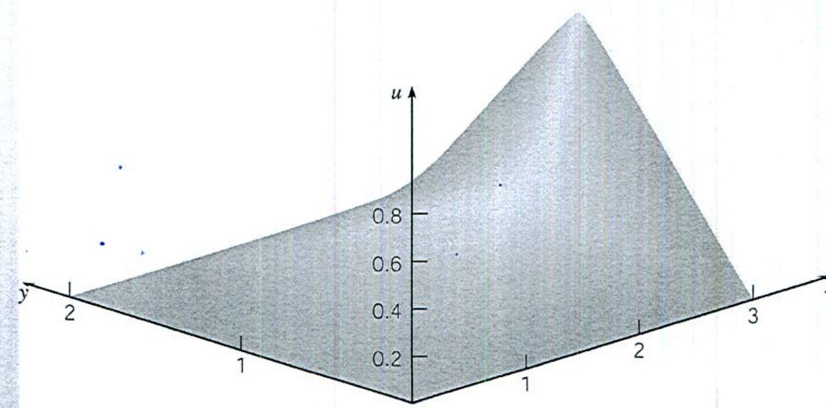
Example 1: Solution (2 of 2)

- The solution to our Dirichlet problem is

$$u(x,\cancel{t}\,y) = \sum_{n=1}^{\infty} \frac{8\sin(n\pi/2)}{n^2\pi^2 \sinh(3n\pi/2)} \sin\!h(n\pi x/2)\,\cancel{\cos}\,\sin(n\pi \cancel{t}\,y/2)$$

- A plot of $u(x,y)$ is given below right, along with a contour plot showing level curves of $u(x,y)$ on the left.





* $u_{xx} + u_{yy} = 0$

$u(x,0) = 0 \quad , \quad u(x,b) = 0$

$u(0,y) = f(y) \quad , \quad u(a,y) = 0$

$u(x,y) = X(x)\, Y(y)$

$\Rightarrow \quad \cancel{X''(x) + \lambda X(x) = 0 \quad , \quad X(0) = 0 \quad , \quad X(b}$

(i) $\quad Y''(y) + \lambda Y = 0 \quad , \quad Y(0) = 0 \quad , \quad Y(b) = 0$

$\Rightarrow \quad \lambda_n = \left(\frac{n\pi}{b}\right)^2 \qquad Y_n(y) = \sin \frac{n\pi y}{b}.$

(ii) $\quad X''(x) - \lambda_n X = 0 \quad , \quad X(a) = 0$

$\Rightarrow \quad X(x) = c_1 \sinh \frac{n\pi}{b}(x-a) + c_2 \cosh \frac{n\pi}{b}(x-a)$

$0 = X(a) = C_1 \cdot 0 + C_2 \cdot 1 \Rightarrow C_2 = 0$

$\Rightarrow \quad u(x,y) = \sum_{n=1}^{\infty} C_n \cancel{\cosh} \sinh \frac{n\pi(x-a)}{b} \sin \frac{n\pi y}{b}.$

$C_n = \frac{2}{b} \frac{1}{\sinh \frac{n\pi(x-a)}{b}} \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy.$

Dirichlet Problem on a Circle (1 of 8)

$u_{xx} + u_{yy} = 0$ in $\{r < a\}$.

- Consider problem of solving a Laplace's equation on a circular region $r < a$ subject to the boundary condition

polar coordinates:

$$u(a,\theta) = f(\theta), \quad 0 \le \theta < 2\pi$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

where f is a given function. See figure below.

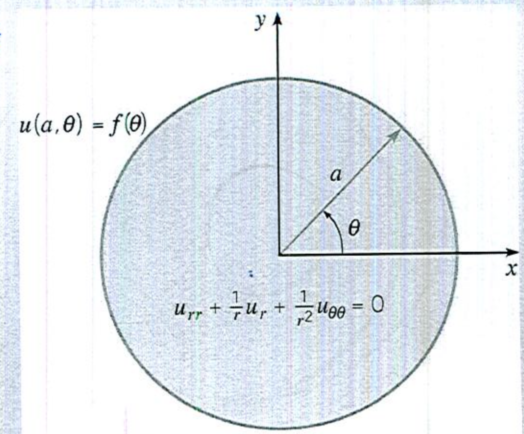
- In polar coordinates, Laplace's equation has the form

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$r < a$

$0 \le \theta < 2\pi$

- We require that $u(r,\theta)$ be periodic in θ with period 2π, and that $u(r,\theta)$ be bounded for $r \le a$.



Separation of Variables Method

- We begin by assuming

$$u(r,\theta) = R(r)\Theta(\theta)$$

- Substituting this into our differential equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

we obtain

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \quad \Rightarrow \left(R''(r) + \frac{1}{r}R'(r)\right)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) =$$

or

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda \Rightarrow \begin{array}{l} r^2R'' + rR' - \lambda R = 0 \\ \Theta'' + \lambda\Theta = 0 \end{array}$$

, Θ periodic in θ with period 2π.

where λ is a constant.

$\Theta'' + \lambda \Theta = 0$ Θ periodic.

Equations for $\lambda < 0$, $\lambda = 0$

- Since $u(r,t)$ is periodic in θ with period 2π, it can be shown that λ is real. We consider the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.
- If $\lambda < 0$, let $\lambda = -\mu^2$ where $\mu > 0$. Then

$$\Theta'' - \mu^2 \Theta = 0 \;\Rightarrow\; \Theta = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta}$$

- Thus $\Theta(\theta)$ periodic only if $c_1 = c_2 = 0$; hence λ is not negative.
- If $\lambda = 0$, then the solution of $\Theta = 0$ is $\Theta = c_1 + c_2\theta$.
- Thus $\Theta(\theta)$ periodic only if $c_2 = 0$; hence $\Theta(\theta)$ is a constant.
- Further, the corresponding equation for R is the Euler equation

$$r^2 R'' + rR' = 0 \;\Rightarrow\; R(r) = k_1 + k_2 \ln r$$

- Since $u(r,t)$ bounded for $r \le a$, $k_2 = 0$ and thus $R(r)$ is constant.
- Hence the solution $u(r,\theta)$ is constant for $\lambda = 0$.

Equations for $\lambda > 0$

- If $\lambda > 0$, let $\lambda = \mu^2$ where $\mu > 0$. Then

$$\Theta'' + \mu^2 \Theta = 0 \;\Rightarrow\; \Theta = c_1 \sin(\mu\theta) + c_2 \cos(\mu\theta)$$

- Thus $\Theta(\theta)$ periodic with period 2π only if $\mu = n$, where n is a positive integer.
- Further, the corresponding equation for R is the Euler equation

$$r^2 R'' + rR' + \mu^2 R = 0 \;\Rightarrow\; R(r) = k_1 r^{\mu} + k_2 r^{-\mu}$$

- Since $u(r,t)$ bounded for $r \le a$, $k_2 = 0$ and thus

$$R(r) = k_1 r^{\mu}$$

- It follows that in this case the solutions take the form

$$u_n(r,\theta) = \underline{r^n \cos n\theta}, \quad v_n(r,\theta) = \underline{r^n \sin n\theta}, \quad n = 1, 2, \ldots$$

Look for $R(r) = r^{\lambda} \Rightarrow \cancel{\mu^2 + \mu}\ \lambda^2 - \mu^2 = 0 \Rightarrow \lambda = \pm\mu.$

Fundamental Solutions

- Thus the fundamental solutions of

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

are, for $n = 1, 2, \ldots,$

$$u_0(r,\theta) = 1,\ u_n(r,\theta) = r^n \cos n\theta,\quad v_n(r,\theta) = r^n \sin n\theta$$

- In the usual way, we assume that

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n \left(c_n \cos n\theta + k_n \sin n\theta\right)$$

where c_n and k_n are chosen to satisfy the boundary condition

$$u(a,\theta) = f(\theta),\quad 0 \le \theta < 2\pi$$

Boundary Condition

- Thus

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n \left(c_n \cos n\theta + k_n \sin n\theta\right)$$

where c_n and k_n are chosen to satisfy the boundary condition

$$u(a,\theta) = f(\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n \left(c_n \cos n\theta + k_n \sin n\theta\right), \;\; 0 \le \theta < 2\pi$$

- The function f may be extended outside the interval $0 \le \theta < 2\pi$ so that it is periodic with period 2π, and therefore has a Fourier series of the form above.
- We can therefore compute the coefficients c_n and k_n using the Euler-Fourier formulas.

Coefficients

- Since the periodic extension of f has period 2π, we may compute the Fourier coefficients by integrating over any period of the function.
- In particular, it is convenient to choose $(0, 2\pi)$.
- Thus for

$$f(\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n \left(c_n \cos n\theta + k_n \sin n\theta\right),$$

we have

$$a^n c_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta, \quad n = 0, 1, 2, \ldots$$

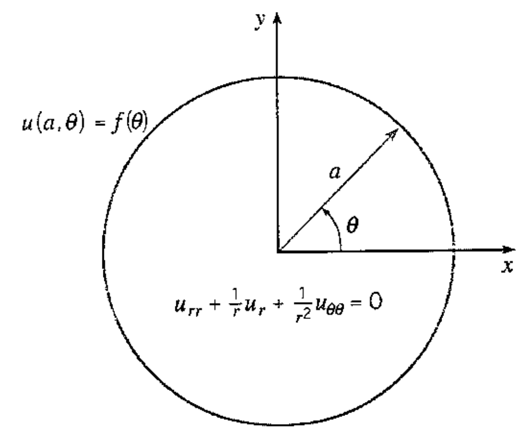
$$a^n k_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta, \quad n = 1, 2, 3, \ldots$$

Solution (8 of 8)

- Therefore the solution to the boundary value problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$

$$u(a,\theta) = f(\theta), \quad 0 \le \theta < 2\pi$$



is given by

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n \left(c_n \cos n\theta + k_n \sin n\theta\right),$$

where

$$c_n = \frac{1}{a^n \pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta, \quad k_n = \frac{1}{a^n \pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta$$

- A full Fourier series is required here, as the boundary data were given on $0 \le \theta < 2\pi$ and have period 2π.