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On fully decoupled MSAV schemes for the Cahn–Hilliard–Navier–Stokes model of two-phase incompressible flows

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We construct first- and second-order time discretization schemes for the Cahn–Hilliard–Navier–Stokes system based on the multiple scalar auxiliary variables (MSAV) approach for gradient systems and (rotational) pressure-correction for Navier–Stokes equations. These schemes are linear, fully decoupled, unconditionally energy stable, and only require solving a sequence of elliptic equations with constant coefficients at each time step. We carry out a rigorous error analysis for the first-order scheme, establishing optimal convergence rate for all relevant functions in different norms. We also provide numerical experiments to verify our theoretical results.

Keywords: Cahn–Hilliard–Navier–Stokes; multiple scalar auxiliary variables (MSAV); fully decoupled; energy stability; error estimates.

AMS Subject Classification 2020: 35G25, 65M12, 65M15, 65Z05, 76D05

1. Introduction

We consider in this paper the construction and analysis of efficient time discretization schemes for the following Cahn–Hilliard–Navier–Stokes system:

$$\frac{\partial \phi}{\partial t} + (\mathbf{u} \cdot \nabla)\phi = M\Delta\mu \quad \text{in } \Omega \times J, \tag{1.1a}$$

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$$\mu = -\lambda\Delta\phi + \lambda G'(\phi) \quad \text{in } \Omega \times J, \tag{1.1b}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mu\nabla\phi \quad \text{in } \Omega \times J, \tag{1.1c}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \times J, \tag{1.1d}$$

$$\frac{\partial \phi}{\partial \mathbf{n}} = \frac{\partial \mu}{\partial \mathbf{n}} = 0, \ \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times J, \tag{1.1e}$$

where $G(\phi) = \frac{1}{4\epsilon^2}(1-\phi^2)^2$ with ϵ representing the interfacial width, $M > 0$ is the mobility constant, $\lambda > 0$ is the mixing coefficient, $\nu > 0$ is the fluid viscosity. Ω is a bounded domain in $\mathbb{R}^2$ and $J = (0, T]$. The unknowns are the velocity $\mathbf{u}$, the pressure p, the phase function ϕ and the chemical potential μ. Here, we set $\mathbf{u}\cdot\nabla = u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y}$. We refer to Refs. 12, 16 and 22 for its physical interpretation and derivation as a phase-field model for the incompressible two-phase flow with matching density (set to be $\rho_0 = 1$ for simplicity), and to Ref. 1 for its mathematical analysis. The above system satisfies the following energy dissipation law:

$$\frac{dE(\phi,\mathbf{u})}{dt} = -M\|\nabla\mu\|^2 - \nu\|\nabla\mathbf{u}\|^2 \quad \text{with } E(\phi,\mathbf{u}) = \int_\Omega \left\{\frac{1}{2}|\mathbf{u}|^2 + \frac{\lambda}{2}|\nabla\phi|^2 + \lambda G(\phi)\right\} d\mathbf{x}. \tag{1.2}$$

For nonlinear dissipative systems such as the Navier–Stokes equation, Cahn–Hilliard equation and Cahn–Hilliard–Navier–Stokes system (1.1), it is important that numerical schemes preserve a dissipative energy law at the discrete level. Various energy stable numerical methods have been proposed in the last few decades for Navier–Stokes equations and for Cahn–Hilliard equations. A main difficulty in solving the Navier–Stokes equation is the coupling of velocity and pressure by the incompressible condition $\nabla\cdot\mathbf{u} = 0$. A popular strategy is to use a projection type method pioneered by Chorin and Temam in late 1960s[3,30] which decouples the computation of pressure and velocity, we can refer to Ref. 9 for a review on various projection type methods for Navier–Stokes equations. The main issue in dealing with the Cahn–Hilliard equation is how to treat the nonlinear term effectively so that the resulting discrete system can be efficiently solved while being energy stable. Popular approaches include the convex splitting,[6] stabilized semi-implicit,[26] invariant energy quadratization (IEQ),[33] and scalar auxiliary variable (SAV).[25] We refer to Ref. 5 (see also Ref. 29) for an up-to-date review on various methods for gradient flows which include in particular the Cahn–Hilliard equation.

On the other hand, it is much more challenging to develop efficient numerical schemes and to carry out corresponding error analysis for phase-field models such as (1.1) coupling Navier–Stokes equations and Cahn–Hilliard equations. The system (1.1) is a highly coupled nonlinear system whose dissipation law (1.2) relies on delicate cancellations of various nonlinear interactions. Usually, energy stable schemes for (1.1) are constructed using fully or weakly coupled fully implicit or partially implicit time discretization. Feng *et al.*[7] considered fully coupled first-order-in-time implicit semi-discrete and fully discrete finite element schemes and established their convergence results. Shen and Yang constructed a sequence of weakly coupled[27] and full decoupled,[28] linear, first-order unconditionally energy stable schemes in time discretization for two-phase incompressible flows with same or different densities and viscosities with a modified double-well potential. Grün[8] established an abstract convergence result of a fully discrete implicit scheme for diffuse interface models of two-phase incompressible fluids with different densities. Han and Wang[14] constructed a coupled second-order energy stable scheme for the Cahn–Hilliard–Navier–Stokes system based on convex splitting for the Cahn–Hilliard equation, a related fully discrete scheme is constructed in Ref. 4 where second-order convergence in time is established. Han *et al.*[13] developed a class of second-order energy stable schemes based on the IEQ approach. Recently, in Ref. 18, we constructed a second-order weakly-coupled, linear, energy stable SAV-MAC scheme for the Cahn–Hilliard–Navier–Stokes equations, and established second-order convergence both in time and space for the simpler Cahn–Hilliard–Stokes equations. Note that in all these works, a coupled linear or nonlinear system with variable coefficients has to be solved at each time step. We refer to the aforementioned papers for the references therein for other related work on this subject.

We would like to point out that Yang and Dong[34] developed linear and unconditionally energy-stable schemes for a more complicated phase-field model of two-phase incompressible flow with variable density, but the velocity and pressure are still coupled and it requires solving a nonlinear algebraic equation at each time step. To the best of our knowledge, despite a large number of works devoted to the construction and analysis for the Cahn–Hilliard–Navier–Stokes system (1.1), there is still no fully decoupled, linear, second-order-in-time, unconditionally energy stable scheme, and there is no error analysis for any fully decoupled schemes for (1.1) as all previous analyses are for schemes which are either fully coupled or weakly coupled. In particular, it is highly nontrivial to establish error estimates for fully decoupled linear schemes due to additional difficulties which arise from explicit treatment of nonlinear terms and the extra splitting error due to the decoupling of pressure from velocity.

The main purposes of this work are (i) to construct first- and second-order fully decoupled, linear and unconditionally energy stable schemes for (1.1), and (ii) to carry out a rigorous error analysis. By using a combination of techniques in the multiple SAV approach,[2] pressure-correction and rotational pressure-correction[11] and a special SAV approach for the Navier–Stokes equation,[20] we are finally able to construct a fully decoupled, linear, second-order-in-time, unconditionally energy stable scheme for (1.1). Furthermore, the schemes we constructed do not involve a nonlinear algebraic equation as in Refs. 17 and 21 and lead to bounds including the kinetic energy $\frac{1}{2}\|u\|^2$ rather than a positive SAV constant as an approximation to the kinetic energy as in Refs. 17 and 21. This turns out to be crucial in the error analysis. More precisely, the work presented in this paper for (1.1) is unique in the following aspects: (i) we construct fully decoupled, unconditionally energy stable, first- and second-order linear schemes which only require solving a sequence of elliptic equations with constant coefficients at each time; (ii) we establish rigorous first-order error estimates in time for all relevant functions in different norms without using an induction argument which often requires restriction on the time step. The key property is that our schemes lead to uniform bounds on the kinetic energy $\frac{1}{2}\|u\|^2$. We believe that our second-order scheme is the first fully decoupled, linear, second-order-in-time, unconditionally energy stable scheme for (1.1), and our error analysis for the first-order scheme is the first for any linear and fully decoupled schemes for (1.1) with explicit treatment of all nonlinear terms.

This paper is organized as follows. In Sec. 2, we describe some notations and useful inequalities. In Sec. 3, we construct the fully decoupled multiple scalar auxiliary variables (MSAV) schemes, prove their unconditional energy stability, and describe an efficient procedure for their implementation. In Sec. 4, we carry out error estimates for the first-order MSAV scheme for all functions except the pressure. In Sec. 5, we present numerical experiments to verify the accuracy of the theoretical results. The error estimate for the pressure is derived in the appendix.

2. Preliminaries

We first introduce some standard notations. Let $L^m(\Omega)$ be the standard Banach space with norm

$$\|v\|_{L^m(\Omega)} = \left(\int_\Omega |v|^m d\Omega\right)^{1/m}.$$

For simplicity, let

$$(f,g) = (f,g)_{L^2(\Omega)} = \int_\Omega fg d\Omega$$

denote the $L^2(\Omega)$ inner product. For the case $p=\infty$, set $\|v\|_\infty = \|v\|_{L^\infty(\Omega)} = \text{ess sup}\{|f(x)| : x \in \Omega\}$. And $W^{k,p}(\Omega)$ be the standard Sobolev space

$$W^{k,p}(\Omega) = \{g : \|g\|_{W_p^k(\Omega)} < \infty\},$$

where

$$\|g\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^\alpha g\|_{L^p(\Omega)}^p\right)^{1/p}, \tag{2.1}$$

in the case $1 \le p < \infty$, and in the case $p = \infty$,

$$\|g\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha|\le k} \|D^\alpha g\|_{L^\infty(\Omega)}.$$

For simplicity, we set $H^k(\Omega) = W^{k,2}(\Omega)$ and $\|f\|_k = \|f\|_{H^k(\Omega)}$.

By using Poincaré inequality, we have

$$\|\mathbf{v}\| \leq c_1\|\nabla \mathbf{v}\|, \quad \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{2.2}$$

where c_1 is a positive constant depending only on Ω and

$$\mathbf{H}_0^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}|_\Gamma = 0\}.$$

Define

$$\mathbf{H} = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathrm{div}\mathbf{v} = 0, \mathbf{v}\cdot\mathbf{n}|_\Gamma = 0\}, \quad \mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \mathrm{div}\mathbf{v} = 0\},$$

and the trilinear form $b(\cdot,\cdot,\cdot)$ by

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}d\mathbf{x}.$$

We can easily observe that the trilinear form $b(\cdot,\cdot,\cdot)$ is skew-symmetric with respect to its last two arguments, i.e.

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}), \quad \forall\,\mathbf{u} \in \mathbf{H}, \quad \mathbf{v},\ \mathbf{w} \in \mathbf{H}_0^1(\Omega), \tag{2.3}$$

and

$$b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \quad \forall\,\mathbf{u} \in \mathbf{H}, \quad \mathbf{v} \in \mathbf{H}_0^1(\Omega). \tag{2.4}$$

By using a combination of integration by parts, Holder's inequality and Sobolev inequalities, we have Ref. 31 for $d \leq 2$

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) \leq \begin{cases} c_2\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1, & \forall\,\mathbf{u},\mathbf{v} \in \mathbf{H}, \quad \mathbf{w} \in \mathbf{H}_0^1(\Omega), \\ c_2\|\mathbf{u}\|_2\|\mathbf{v}\|\|\mathbf{w}\|_1, & \forall\,\mathbf{u} \in \mathbf{H}^2(\Omega)\cap\mathbf{H}, \quad \mathbf{v} \in \mathbf{H}, \quad \mathbf{w} \in \mathbf{H}_0^1(\Omega), \\ c_2\|\mathbf{u}\|_2\|\mathbf{v}\|_1\|\mathbf{w}\|, & \forall\,\mathbf{u} \in \mathbf{H}^2(\Omega)\cap\mathbf{H}, \quad \mathbf{v} \in \mathbf{H}, \quad \mathbf{w} \in \mathbf{H}_0^1(\Omega), \\ c_2\|\mathbf{u}\|_1\|\mathbf{v}\|_2\|\mathbf{w}\|, & \forall\,\mathbf{v} \in \mathbf{H}^2(\Omega)\cap\mathbf{H}, \quad \mathbf{u} \in \mathbf{H}, \quad \mathbf{w} \in \mathbf{H}_0^1(\Omega), \\ c_2\|\mathbf{u}\|\|\mathbf{v}\|_2\|\mathbf{w}\|_1, & \forall\,\mathbf{v} \in \mathbf{H}^2(\Omega)\cap\mathbf{H}, \quad \mathbf{u} \in \mathbf{H}, \quad \mathbf{w} \in \mathbf{H}_0^1(\Omega), \\ c_2\|\mathbf{u}\|_1^{1/2}\|\mathbf{u}\|^{1/2}\|\mathbf{v}\|_1^{1/2} & \\ \quad \|\mathbf{v}\|^{1/2}\|\mathbf{w}\|_1, & \forall\,\mathbf{u},\mathbf{v} \in \mathbf{H}, \quad \mathbf{w} \in \mathbf{H}_0^1(\Omega), \end{cases} \tag{2.5}$$

where c_2 is a positive constant depending only on Ω.

Let $P_\mathbf{H}$ be the projection operator in $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$. We have (cf. (1.47) in Ref. 32)

$$\|P_\mathbf{H} u\|_1 \leq C(\Omega)\|u\|_1, \quad \forall\, u \in \mathbf{H}^1(\Omega). \tag{2.6}$$

We will frequently use the following discrete version of the Grönwall lemma[15,23]:

Lemma 2.1. *Let a_k, b_k, c_k, d_k, γ_k, Δt_k be nonnegative real numbers such that*

$$a_{k+1} - a_k + b_{k+1}\Delta t_{k+1} + c_{k+1}\Delta t_{k+1} - c_k\Delta t_k \leq a_k d_k \Delta t_k + \gamma_{k+1}\Delta t_{k+1} \tag{2.7}$$

for all $0 \le k \le m$. *Then*

$$a_{m+1} + \sum_{k=0}^{m+1} b_k \Delta t_k \le \exp\left(\sum_{k=0}^{m} d_k \Delta t_k\right) \left\{ a_0 + (b_0 + c_0)\Delta t_0 + \sum_{k=1}^{m+1} \gamma_k \Delta t_k \right\}. \tag{2.8}$$

Throughout the paper, we use C, with or without subscript, to denote a positive constant, independent of discretization parameters, which could have different values at different places.

3. The MSAV Schemes

In this section, we first reformulate the Cahn–Hilliad–Navier–Stokes system into an equivalent system with MSAV. Then, we construct first- and second-order fully decoupled semi-discrete MSAV schemes, present a detailed procedure to efficiently implement them, and prove that they are unconditionally energy stable.

3.1. *MSAV reformulation*

Let $\gamma > 0$ be a positive constant, $F(\phi) = G(\phi) - \frac{\gamma}{2}\phi^2$ and $E_1(\phi) = \int_\Omega F(\phi)d\mathbf{x}$. Here, the term $\frac{\gamma}{2}\phi^2$ is introduced to simplify the analysis (cf. Ref. 24). We introduce the following two scalar auxiliary variables:

$$r(t) = \sqrt{E_1(\phi) + \delta}, \quad \forall\, \delta > \gamma, \tag{3.1a}$$

$$q(t) = \exp\left(-\frac{t}{T}\right), \tag{3.1b}$$

where $\epsilon \ll 1$, and reformulate the system (1.1) as

$$\frac{\partial \phi}{\partial t} + \frac{r}{\sqrt{E_1(\phi)+\delta}}(\mathbf{u}\cdot\nabla)\phi = M\Delta\mu \quad \text{in } \Omega \times J, \tag{3.2a}$$

$$\mu = -\lambda\Delta\phi + \lambda\gamma\phi + \lambda\frac{r}{\sqrt{E_1(\phi)+\delta}}F'(\phi) \quad \text{in } \Omega \times J, \tag{3.2b}$$

$$\frac{\mathrm{dr}}{\mathrm{dt}} = \frac{1}{2\sqrt{E_1(\phi)+\delta}}\int_\Omega F'(\phi)\frac{\partial\phi}{\partial t}d\mathbf{x} \quad \text{in } \Omega \times J, \tag{3.2c}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \exp\left(\frac{t}{T}\right) q(t)\mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \frac{r}{\sqrt{E_1(\phi)+\delta}}\mu\nabla\phi \quad \text{in } \Omega \times J, \tag{3.2d}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \times J, \tag{3.2e}$$

$$\frac{\mathrm{dq}}{\mathrm{dt}} = -\frac{1}{T}q + \exp\left(\frac{t}{T}\right)\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}d\mathbf{x} \quad \text{in } \Omega \times J. \tag{3.2f}$$

Since $\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}d\mathbf{x} = 0$, it is easy to see that, with $r(0) = \sqrt{E_1(\phi|_{t=0}) + \delta}$ and $q(0) = 1$, the above system is equivalent to the original system. Taking the inner products of (3.2a) with μ, (3.2b) with $\frac{\partial \phi}{\partial t}$, (3.2d) with $\mathbf{u}$, and multiplying (3.2c) with $2r$, summing up the results, we obtain the original energy law (1.2). Furthermore, if we take the inner product of (3.2f) with q and add it to the previous expression, we obtain an equivalent dissipation law:

$$\frac{d\tilde{E}(\phi, \mathbf{u}, r)}{dt} = -M\|\nabla \mu\|^2 - \nu\|\nabla \mathbf{u}\|^2 - \frac{1}{T}q^2, \tag{3.3}$$

where $\tilde{E}(\phi, \mathbf{u}, r, q) = \int_\Omega \frac{1}{2}\{|\mathbf{u}|^2 + \lambda\gamma\phi^2 + \lambda|\nabla\phi|^2\}d\mathbf{x} + \frac{1}{2}q^2 + \lambda r^2$. We shall construct below efficient numerical schemes for the above system which are energy stable with respect to (3.3).

3.2. *A first-order scheme*

We denote

$$\Delta t = T/N, \quad t^n = n\Delta t, \quad d_t g^n = \frac{g^n - g^{n-1}}{\Delta t}, \quad \text{for } n \le N.$$

Our first-order scheme for (3.2) is as follows: Find $(\phi^{n+1}, \mu^{n+1}, \tilde{\mathbf{u}}^{n+1}, \mathbf{u}^{n+1}, p^{n+1}, r^{n+1}, q^{n+1})$ such that

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} + \frac{r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}(\mathbf{u}^n \cdot \nabla)\phi^n = M\Delta\mu^{n+1}, \tag{3.4}$$

$$\mu^{n+1} = -\lambda\Delta\phi^{n+1} + \lambda\gamma\phi^{n+1} + \lambda\frac{r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}F'(\phi^n); \tag{3.5}$$

$$\begin{aligned}\frac{r^{n+1} - r^n}{\Delta t} &= \frac{1}{2\sqrt{E_1(\phi^n) + \delta}}\left(\left(F'(\phi^n), \frac{\phi^{n+1} - \phi^n}{\Delta t}\right) + \frac{1}{\lambda}(\mu^{n+1}, \mathbf{u}^n \cdot \nabla\phi^n)\right) \\ &\quad - \frac{1}{2\lambda\sqrt{E_1(\phi^n) + \delta}}(\tilde{\mathbf{u}}^{n+1}, \mu^n\nabla\phi^n);\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\frac{\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\Delta t} + \exp\left(\frac{t^{n+1}}{T}\right) q^{n+1}\mathbf{u}^n \cdot \nabla\mathbf{u}^n - \nu\Delta\tilde{\mathbf{u}}^{n+1} + \nabla p^n \\ &\qquad = \frac{r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}\mu^n\nabla\phi^n, \quad \tilde{\mathbf{u}}^{n+1}|_{\partial\Omega} = 0;\end{aligned} \tag{3.7}$$

$$\nabla \cdot \mathbf{u}^{n+1} = 0, \quad \mathbf{u}^{n+1} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{3.8}$$

$$\frac{\mathbf{u}^{n+1} - \tilde{\mathbf{u}}^{n+1}}{\Delta t} + \nabla(p^{n+1} - p^n) = 0; \tag{3.9}$$

$$\frac{q^{n+1} - q^n}{\Delta t} = -\frac{1}{T}q^{n+1} + \exp\left(\frac{t^{n+1}}{T}\right)(\mathbf{u}^n \cdot \nabla\mathbf{u}^n, \tilde{\mathbf{u}}^{n+1}). \tag{3.10}$$

Note that we added the terms $(\mu^{n+1}, \mathbf{u}^n \cdot \nabla\phi^n) - (\tilde{\mathbf{u}}^{n+1}, \mu^n\nabla\phi^n)$ in (3.6) which is a first-order approximation to $(\mu, \mathbf{u}\nabla\phi) - (\mathbf{u}, \mu\nabla\phi) = 0$. On the other hand, (3.7)-(3.9) is a first-order pressure-correction scheme[9] for (3.2d)–(3.2e). Hence, the above scheme is first-order consistent to (3.2).

Remark 3.1. There are two main differences between the current scheme and the scheme in Ref. 18 (and other schemes for (1.1)):

- We employ a pressure-correction technique to decouple the computation of pressure and velocity.
- We introduced two SAVs here instead of one in Ref. 18. The second SAV, $q(t)$, allows us to totally decouple the numerical scheme, as opposed to weakly coupled in Ref. 18, as well as avoiding solving a nonlinear algebraic equation at each time step, which presents great challenge in establishing well-posedness and error estimates of the scheme.

3.2.1. *Efficient implementation*

We observe that the above scheme is linear but coupled. A remarkable property is that the scheme can be decoupled as we show below. Denote

$$\xi_1^{n+1} = \frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}, \quad \xi_2^{n+1} = \exp\left(\frac{t^{n+1}}{T}\right) q^{n+1}, \tag{3.11}$$

and set

$$\begin{cases} \phi^{n+1} = \phi_0^{n+1} + \xi_1^{n+1}\phi_1^{n+1} + \xi_2^{n+1}\phi_2^{n+1}, \\ \mu^{n+1} = \mu_0^{n+1} + \xi_1^{n+1}\mu_1^{n+1} + \xi_2^{n+1}\mu_2^{n+1}, \\ \tilde{\mathbf{u}}^{n+1} = \tilde{\mathbf{u}}_0^{n+1} + \xi_1^{n+1}\tilde{\mathbf{u}}_1^{n+1} + \xi_2^{n+1}\tilde{\mathbf{u}}_2^{n+1}, \\ \mathbf{u}^{n+1} = \mathbf{u}_0^{n+1} + \xi_1^{n+1}\mathbf{u}_1^{n+1} + \xi_2^{n+1}\mathbf{u}_2^{n+1}, \\ p^{n+1} = p_0^{n+1} + \xi_1^{n+1}p_1^{n+1} + \xi_2^{n+1}p_2^{n+1}. \end{cases} \tag{3.12}$$

Plugging (3.12) in (3.4)–(3.5) and (3.7)–(3.9), and collecting terms without ξ_1^{n+1}, ξ_2^{n+1}, with ξ_1^{n+1} and with ξ_2^{n+1}, respectively, we can obtain ϕ_i^{n+1}, μ_i^{n+1}, $\mathbf{u}_i^{n+1}$, $\tilde{\mathbf{u}}_i^{n+1}$ and p_i^{n+1} $(i = 0, 1, 2)$ as follows:

Step 1: Find $(\phi_i^{n+1}, \mu_i^{n+1})$ $(i = 0, 1, 2)$ such that

$$\phi_0^{n+1} - \phi^n = M\Delta t\Delta\mu_0^{n+1}, \quad \mu_0^{n+1} = -\lambda\Delta\phi_0^{n+1} + \lambda\gamma\phi_0^{n+1}, \tag{3.13}$$

$$\phi_1^{n+1} + \Delta t(\mathbf{u}^n\cdot\nabla)\phi^n = M\Delta t\Delta\mu_1^{n+1}, \quad \mu_1^{n+1} = -\lambda\Delta\phi_1^{n+1} + \lambda\gamma\phi_1^{n+1} + \lambda F'(\phi^n), \tag{3.14}$$

$$\phi_2^{n+1} = M\Delta t\Delta\mu_2^{n+1}, \quad \mu_2^{n+1} = -\lambda\Delta\phi_2^{n+1} + \lambda\gamma\phi_2^{n+1}. \tag{3.15}$$

We derive immediately from the last relation that $\phi_2^{n+1} = 0$, $\mu_2^{n+1} = 0$. On the other hand, (3.13) (respectively, (3.14)) is a coupled second-order system with constant coefficients in the same form as a simple semi-implicit scheme for the Cahn–Hilliard equation.

Step 2: Find $\tilde{\mathbf{u}}_i^{n+1} (i = 0, 1, 2)$ such that

$$\tilde{\mathbf{u}}_0^{n+1} - \mathbf{u}^n - \nu\Delta t\Delta\tilde{\mathbf{u}}_0^{n+1} + \Delta t\nabla p^n = 0, \quad \tilde{\mathbf{u}}_0^{n+1}|_{\partial\Omega} = 0, \tag{3.16}$$

$$\tilde{\mathbf{u}}_1^{n+1} - \nu\Delta t\Delta\tilde{\mathbf{u}}_1^{n+1} = \Delta t\mu^n\nabla\phi^n, \quad \tilde{\mathbf{u}}_1^{n+1}|_{\partial\Omega} = 0, \tag{3.17}$$

$$\tilde{\mathbf{u}}_2^{n+1} + \Delta t\mathbf{u}^n\cdot\nabla\mathbf{u}^n - \nu\Delta t\Delta\tilde{\mathbf{u}}_2^{n+1} = 0, \quad \tilde{\mathbf{u}}_2^{n+1}|_{\partial\Omega} = 0. \tag{3.18}$$

The above three systems are decoupled second-order equations with same constant coefficients.

Step 3: Find $(\mathbf{u}_i^{n+1}, p_i^{n+1})(i = 0, 1, 2)$ such that

$$\mathbf{u}_0^{n+1} - \tilde{\mathbf{u}}_0^{n+1} + \Delta t\nabla(p_0^{n+1} - p^n) = 0, \quad \nabla\cdot\mathbf{u}_0^{n+1} = 0, \quad \mathbf{u}_0^{n+1}\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{3.19}$$

$$\mathbf{u}_i^{n+1} - \tilde{\mathbf{u}}_i^{n+1} + \Delta t\nabla p_i^{n+1} = 0, \quad \nabla\cdot\mathbf{u}_i^{n+1} = 0, \quad \mathbf{u}_i^{n+1}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad i = 1, 2. \tag{3.20}$$

The above systems correspond to the projection step in the pressure-correction scheme for Navier–Stokes equations. By taking the divergence operator on each of the above system, we find that $p_i^{n+1}(i = 0, 1, 2)$ can be determined by solving a Poisson equation with homogeneous boundary conditions,[9] and then $\mathbf{u}_i^{n+1}(i = 0, 1, 2)$ can be obtained explicitly.

Once ϕ_i^{n+1}, μ_i^{n+1}, $\mathbf{u}_i^{n+1}$, $\tilde{\mathbf{u}}_i^{n+1}$ and p_i^{n+1} $(i = 0, 1, 2)$ are known, we are now in position to determine ξ_1^{n+1} and ξ_2^{n+1}. From (3.6) and (3.10), we find that ξ_1^{n+1} and ξ_2^{n+1} can be explicitly determined by solving the following 2×2 linear algebraic system:

$$\begin{aligned} A_1\xi_1 + A_2\xi_2 &= A_0, \\ B_1\xi_1 + B_2\xi_2 &= B_0, \end{aligned} \tag{3.21}$$

where

$$\begin{cases} A_0 = \dfrac{r^n}{\Delta t} + \dfrac{1}{2\sqrt{E_1(\phi^n)+\delta}}\left(\left(F'(\phi^n), \dfrac{\phi_0^{n+1}-\phi^n}{\Delta t}\right) + \dfrac{1}{\lambda}(\mu_0^{n+1}, \mathbf{u}^n\cdot\nabla\phi^n)\right. \\ \qquad \left. - \dfrac{1}{\lambda}(\tilde{\mathbf{u}}_0^{n+1}, \mu^n\nabla\phi^n)\right), \\ A_1 = \dfrac{\sqrt{E_1(\phi^n)+\delta}}{\Delta t} - \dfrac{1}{2\sqrt{E_1(\phi^n)+\delta}}\left(\left(F'(\phi^n), \dfrac{\phi_1^{n+1}}{\Delta t}\right)\right. \\ \qquad \left. + \dfrac{1}{\lambda}(\mu_1^{n+1}, \mathbf{u}^n\cdot\nabla\phi^n) - \dfrac{1}{\lambda}(\tilde{\mathbf{u}}_1^{n+1}, \mu^n\nabla\phi^n)\right), \\ A_2 = \dfrac{1}{2\lambda\sqrt{E_1(\phi^n)+\delta}}(\tilde{\mathbf{u}}_2^{n+1}, \mu^n\nabla\phi^n), \\ B_0 = \dfrac{q^n}{\Delta t} + \exp\left(\dfrac{t^{n+1}}{T}\right)(\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{\mathbf{u}}_0^{n+1}), \\ B_1 = -\exp\left(\dfrac{t^{n+1}}{T}\right)(\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{\mathbf{u}}_1^{n+1}), \\ B_2 = \dfrac{\exp(-\frac{t^{n+1}}{T})}{\Delta t} + \dfrac{\exp(-\frac{t^{n+1}}{T})}{T} - \exp\left(\dfrac{t^{n+1}}{T}\right)(\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{\mathbf{u}}_2^{n+1}). \end{cases} \tag{3.22}$$

In summary, at each time step, we only need to solve two coupled second-order systems with the same constant coefficients in (3.13)–(3.15), and three Poisson-type equations in (3.16)–(3.18), and three Poisson equations (3.19)–(3.20). Finally, we can determine $\xi_i (i = 1, 2)$ by solving a 2×2 linear algebraic system (3.21) with negligible computational cost. Hence, this scheme is very efficient and easy to implement.

3.2.2. *Unique solvability and energy stability*

We show first that the 2×2 linear system (3.21) is uniquely solvable.

Lemma 3.1. *For the* 2×2 *linear system* (3.21), *we have* $A_2B_1 - A_1B_2 \neq 0$.

Proof. Taking the inner product of the first equation in (3.14), the second equation in (3.14) and (3.17) with μ_1^{n+1}, ϕ_1^{n+1} and $\tilde{\mathbf{u}}_1^{n+1}$, we have

$$\Delta t\big((\mathbf{u}^n \cdot \nabla)\phi^n, \mu_1^{n+1}\big) = -(\phi_1^{n+1}, \mu_1^{n+1}) - M\Delta t\|\nabla \mu_1^{n+1}\|^2, \tag{3.23}$$

$$\lambda\big(F'(\phi^n), \phi_1^{n+1}\big) = (\mu_1^{n+1}, \phi_1^{n+1}) - \lambda\|\nabla \phi_1^{n+1}\|^2 - \lambda\gamma\|\phi_1^{n+1}\|^2, \tag{3.24}$$

$$\Delta t(\mu^n \nabla \phi^n, \tilde{\mathbf{u}}_1^{n+1}) = \|\tilde{\mathbf{u}}_1^{n+1}\|^2 + \nu\Delta t\|\nabla \tilde{\mathbf{u}}_1^{n+1}\|^2. \tag{3.25}$$

Thus substituting (3.23)–(3.25) into the formula for A_1 in (3.22), we can obtain

$$\begin{aligned} A_1 = {} & \frac{\sqrt{E_1(\phi^n) + \delta}}{\Delta t} + \frac{1}{2\sqrt{E_1(\phi^n) + \delta}} \left(\frac{1}{\Delta t}\|\nabla \phi_1^{n+1}\|^2 + \frac{\gamma}{\Delta t}\|\phi_1^{n+1}\|^2 \right) \\ & + \frac{1}{2\lambda\sqrt{E_1(\phi^n) + \delta}} \left(M\|\nabla \mu_1^{n+1}\|^2 + \frac{1}{\Delta t}\|\tilde{\mathbf{u}}_1^{n+1}\|^2 + \nu\|\nabla \tilde{\mathbf{u}}_1^{n+1}\|^2 \right). \end{aligned} \tag{3.26}$$

By taking the inner product of (3.17) with $\tilde{\mathbf{u}}_2^{n+1}$, we can also recast the formula for A_2 in (3.22) as

$$A_2 = \frac{1}{2\lambda\sqrt{E_1(\phi^n) + \delta}} \left(\frac{1}{\Delta t}(\tilde{\mathbf{u}}_1^{n+1}, \tilde{\mathbf{u}}_2^{n+1}) + \nu(\nabla \tilde{\mathbf{u}}_1^{n+1}, \nabla \tilde{\mathbf{u}}_2^{n+1}) \right). \tag{3.27}$$

Similarly by taking the inner product of (3.18) with $\tilde{\mathbf{u}}_1^{n+1}$ and $\tilde{\mathbf{u}}_2^{n+1}$, respectively, we have

$$\Delta t(\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \tilde{\mathbf{u}}_1^{n+1}) = -(\tilde{\mathbf{u}}_1^{n+1}, \tilde{\mathbf{u}}_2^{n+1}) - \nu\Delta t(\nabla \tilde{\mathbf{u}}_1^{n+1}, \nabla \tilde{\mathbf{u}}_2^{n+1}), \tag{3.28}$$

$$\Delta t(\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \tilde{\mathbf{u}}_2^{n+1}) = -\|\tilde{\mathbf{u}}_2^{n+1}\|^2 - \nu\Delta t\|\nabla \tilde{\mathbf{u}}_2^{n+1}\|^2. \tag{3.29}$$

Hence the formulas for B_1 and B_2 in (3.22) can be transformed into

$$B_1 = \exp\left(\frac{t^{n+1}}{T}\right) \left(\frac{1}{\Delta t}(\tilde{\mathbf{u}}_1^{n+1}, \tilde{\mathbf{u}}_2^{n+1}) + \nu(\nabla \tilde{\mathbf{u}}_1^{n+1}, \nabla \tilde{\mathbf{u}}_2^{n+1}) \right), \tag{3.30}$$

$$B_2 = \left(\frac{1}{\Delta t} + \frac{1}{T}\right) \exp\left(-\frac{t^{n+1}}{T}\right) + \exp\left(\frac{t^{n+1}}{T}\right) \left(\frac{1}{\Delta t}\|\tilde{\mathbf{u}}_2^{n+1}\|^2 + \nu\|\nabla \tilde{\mathbf{u}}_2^{n+1}\|^2 \right). \tag{3.31}$$

Since the term

$$\begin{aligned}
A_2B_1 &= \frac{\exp(\frac{t^{n+1}}{T})}{2\lambda\sqrt{E_1(\phi^n)+\delta}}\left(\frac{1}{\Delta t}(\tilde{\mathbf{u}}_1^{n+1},\tilde{\mathbf{u}}_2^{n+1})+\nu(\nabla\tilde{\mathbf{u}}_1^{n+1},\nabla\tilde{\mathbf{u}}_2^{n+1})\right)^2 \\
&\le \frac{\exp(\frac{t^{n+1}}{T})}{2\lambda\sqrt{E_1(\phi^n)+\delta}}\left(\frac{1}{(\Delta t)^2}\|\tilde{\mathbf{u}}_1^{n+1}\|^2\|\tilde{\mathbf{u}}_2^{n+1}\|^2+\frac{2\nu}{\Delta t}\|\tilde{\mathbf{u}}_1^{n+1}\|\|\tilde{\mathbf{u}}_2^{n+1}\|\|\nabla\tilde{\mathbf{u}}_1^{n+1}\|\right. \\
&\quad \left.\times\|\nabla\tilde{\mathbf{u}}_2^{n+1}\|\right)+\frac{\exp(\frac{t^{n+1}}{T})\nu^2}{2\lambda\sqrt{E_1(\phi^n)+\delta}}\|\nabla\tilde{\mathbf{u}}_1^{n+1}\|^2\|\nabla\tilde{\mathbf{u}}_2^{n+1}\|^2 \\
&\le \frac{\exp(\frac{t^{n+1}}{T})}{2\lambda\sqrt{E_1(\phi^n)+\delta}}\left(\frac{1}{\Delta t}\|\tilde{\mathbf{u}}_1^{n+1}\|^2+\nu\|\nabla\tilde{\mathbf{u}}_1^{n+1}\|^2\right) \\
&\quad \times\left(\frac{1}{\Delta t}\|\tilde{\mathbf{u}}_2^{n+1}\|^2+\nu\|\nabla\tilde{\mathbf{u}}_2^{n+1}\|^2\right) \\
&< A_1B_2, \qquad (3.32)
\end{aligned}$$

which implies the desired result. □

Theorem 3.1. *The scheme* (3.4)–(3.10) *admits a unique solution, and is unconditionally energy stable in the sense that*

$$\begin{aligned}
&\tilde{E}^{n+1}(\phi,\boldsymbol{u},r,q)-\tilde{E}^n(\phi,\boldsymbol{u},r,q) \\
&\quad \le -2M\Delta t\|\nabla\mu^{n+1}\|^2-2\nu\Delta t\|\nabla\tilde{\boldsymbol{u}}^{n+1}\|^2-\frac{2\Delta t}{T}|q^{n+1}|^2, \qquad (3.33)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{E}^{n+1}(\phi,\boldsymbol{u},r,q) &= \lambda\|\nabla\phi^{n+1}\|^2+\lambda\gamma\|\phi^{n+1}\|^2+2\lambda|r^{n+1}|^2+\|\boldsymbol{u}^{n+1}\|^2 \\
&\quad +(\Delta t)^2\|\nabla p^{n+1}\|^2+|q^{n+1}|^2. \qquad (3.34)
\end{aligned}$$

Proof. Obviously, $(\phi_i^{n+1},\mu_i^{n+1},\tilde{\mathbf{u}}_i^{n+1},\mathbf{u}_i^{n+1},p_i^{n+1})$ $(i=0,1,2)$ can be uniquely determined by using (3.13)–(3.20). Lemma 3.1 implies that the 2×2 linear system (3.21) has a unique solution. Hence, the scheme (3.4)–(3.10) admits a unique solution.

Next, we prove the energy stability. Taking the inner products of (3.4) with $2\Delta t\mu^{n+1}$, (3.5) with $2(\phi^{n+1}-\phi^n)$, respectively, and multiplying (3.6) with $4\lambda\Delta t r^{n+1}$, we can obtain

$$\begin{aligned}
&\lambda(\|\nabla\phi^{n+1}\|^2-\|\nabla\phi^n\|^2+\|\nabla\phi^{n+1}-\nabla\phi^n\|^2) \\
&\quad +\lambda\gamma(\|\phi^{n+1}\|^2-\|\phi^n\|^2+\|\phi^{n+1}-\phi^n\|^2) \\
&\quad +2\lambda(|r^{n+1}|^2-|r^n|^2+|r^{n+1}-r^n|^2) \\
&= -\frac{2\Delta t r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}(\tilde{\mathbf{u}}^{n+1},\mu^n\nabla\phi^n)-2M\Delta t\|\nabla\mu^{n+1}\|^2. \qquad (3.35)
\end{aligned}$$

Taking the inner product of (3.7) with $2\Delta t\tilde{\mathbf{u}}^{n+1}$ leads to

$$\|\tilde{\mathbf{u}}^{n+1}\|^2 - \|\mathbf{u}^n\|^2 + \|\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n\|^2 + 2\Delta t \exp\left(\frac{t^{n+1}}{T}\right) q^{n+1}(\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \tilde{\mathbf{u}}^{n+1})$$
$$+ 2\nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2 + 2\Delta t(\tilde{\mathbf{u}}^{n+1}, \nabla p^n) = \frac{2\Delta t r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}(\tilde{\mathbf{u}}^{n+1}, \mu^n \nabla\phi^n). \tag{3.36}$$

Multiplying (3.10) with $2\Delta t q^{n+1}$ gives

$$|q^{n+1}|^2 - |q^n|^2 + |q^{n+1} - q^n|^2 + \frac{2\Delta t}{T}|q^{n+1}|^2$$
$$= 2\Delta t \exp\left(\frac{t^{n+1}}{T}\right) q^{n+1}(\mathbf{u}^n \cdot \nabla\mathbf{u}^n, \tilde{\mathbf{u}}^{n+1}). \tag{3.37}$$

Recalling (3.9), we have

$$\mathbf{u}^{n+1} + \Delta t\nabla p^{n+1} = \tilde{\mathbf{u}}^{n+1} + \Delta t\nabla p^n. \tag{3.38}$$

Taking the inner product of (3.38) with itself on both sides and noticing that $(\nabla p^{n+1}, \mathbf{u}^{n+1}) = -(p^{n+1}, \nabla\cdot\mathbf{u}^{n+1}) = 0$, we have

$$\|\mathbf{u}^{n+1}\|^2 + (\Delta t)^2\|\nabla p^{n+1}\|^2 = \|\tilde{\mathbf{u}}^{n+1}\|^2 + 2\Delta t(\nabla p^n, \tilde{\mathbf{u}}^{n+1}) + (\Delta t)^2\|\nabla p^n\|^2. \tag{3.39}$$

Combining (3.36) with (3.37) and (3.39) results in

$$\|\mathbf{u}^{n+1}\|^2 - \|\mathbf{u}^n\|^2 + \|\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n\|^2 + (\Delta t)^2\|\nabla p^{n+1}\|^2 - (\Delta t)^2\|\nabla p^n\|^2$$
$$+ |q^{n+1}|^2 - |q^n|^2 + |q^{n+1} - q^n|^2 + \frac{2\Delta t}{T}|q^{n+1}|^2 + 2\nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2$$
$$= \frac{2\Delta t r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}(\tilde{\mathbf{u}}^{n+1}, \mu^n\nabla\phi^n). \tag{3.40}$$

Thus, we can obtain the desired result by combining (3.40) with (3.35). □

3.3. *A second-order scheme*

By replacing first-order approximations in the scheme (3.4)–(3.10) with second-order approximations, and using particularly the second-order rotational pressure-correction scheme for Navier–Stokes equations, we can obtain a second-order linear MSAV scheme as follows: Find $(\phi^{n+1}, \mu^{n+1}, \tilde{\mathbf{u}}^{n+1}, \mathbf{u}^{n+1}, p^{n+1}, r^{n+1}, q^{n+1})$ such that

$$\frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\Delta t} + \frac{r^{n+1}}{\sqrt{E_1(\bar{\phi}^{n+1}) + \delta}}(\bar{\mathbf{u}}^{n+1}\cdot\nabla)\bar{\phi}^{n+1} = M\Delta\mu^{n+1}, \tag{3.41}$$

$$\mu^{n+1} = -\lambda\Delta\phi^{n+1} + \lambda\gamma\phi^{n+1} + \frac{\lambda r^{n+1}}{\sqrt{E_1(\bar{\phi}^{n+1}) + \delta}}F'(\bar{\phi}^{n+1}); \tag{3.42}$$

$$\frac{3r^{n+1}-4r^n+r^{n-1}}{2\Delta t} = \frac{1}{2\sqrt{E_1(\bar{\phi}^{n+1})+\delta}} \Bigg(\Bigg(F'(\bar{\phi}^{n+1}), \frac{3\phi^{n+1}-4\phi^n+\phi^{n-1}}{2\Delta t} \Bigg) \tag{3.43}$$

$$+ \frac{1}{\lambda}(\mu^{n+1}, \bar{\mathbf{u}}^{n+1}\cdot\nabla\bar{\phi}^{n+1}) - \frac{1}{\lambda}(\tilde{\mathbf{u}}^{n+1}, \bar{\mu}^{n+1}\nabla\bar{\phi}^{n+1}) \Bigg);$$

$$\frac{3\tilde{\mathbf{u}}^{n+1}-4\mathbf{u}^n+\mathbf{u}^{n-1}}{2\Delta t} + \exp\left(\frac{t^{n+1}}{T}\right) q^{n+1}\bar{\mathbf{u}}^{n+1}\cdot\nabla\bar{\mathbf{u}}^{n+1} - \nu\Delta\tilde{\mathbf{u}}^{n+1} + \nabla p^n \tag{3.44}$$

$$= \frac{r^{n+1}}{\sqrt{E_1(\bar{\phi}^{n+1})+\delta}}\bar{\mu}^{n+1}\nabla\bar{\phi}^{n+1}, \quad \tilde{\mathbf{u}}^{n+1}|_{\partial\Omega} = 0;$$

$$\frac{3\mathbf{u}^{n+1}-3\tilde{\mathbf{u}}^{n+1}}{2\Delta t} + \nabla(p^{n+1}-p^n+\nu\nabla\cdot\tilde{\mathbf{u}}^{n+1}) = 0, \tag{3.45}$$

$$\nabla\cdot\mathbf{u}^{n+1} = 0, \quad \mathbf{u}^{n+1}\cdot\mathbf{n}|_{\partial\Omega} = 0; \tag{3.46}$$

$$\frac{3q^{n+1}-4q^n+q^{n-1}}{2\Delta t} = -\frac{1}{T}q^{n+1} + \exp\left(\frac{t^{n+1}}{T}\right)(\bar{\mathbf{u}}^{n+1}\cdot\nabla\bar{\mathbf{u}}^{n+1}, \tilde{\mathbf{u}}^{n+1}); \tag{3.47}$$

where $\bar{g}^{n+1} = 2g^n - g^{n-1}$ for any sequence $\{g^n\}$.

3.3.1. *Efficient implementation*

The above scheme can be efficiently implemented as the first-order scheme by solving a sequence of linear systems with constant coefficients. In fact, plugging (3.12) in (3.41)–(3.42) and (3.44)–(3.46), and collecting terms without ξ_1^{n+1}, ξ_2^{n+1}, with ξ_1^{n+1} and with ξ_2^{n+1}, respectively, we can obtain, for each $i = 0, 1, 2$, linear systems for $(\phi_i^{n+1}, \mu_i^{n+1})$ similar to (3.13)–(3.15), for $\tilde{\mathbf{u}}_i^{n+1}$ similar to (3.16)–(3.18), and for $(\mathbf{u}_i^{n+1}, p_i^{n+1})$, the corresponding linear systems are

$$\begin{aligned} &\mathbf{u}_0^{n+1} - \tilde{\mathbf{u}}_0^{n+1} + \Delta t\nabla(p_0^{n+1} - p^n + \nu\nabla\cdot\tilde{\mathbf{u}}_0^{n+1}) = 0, \\ &\nabla\cdot\mathbf{u}_0^{n+1} = 0, \quad \mathbf{u}_0^{n+1}\cdot\mathbf{n}|_{\partial\Omega} = 0, \end{aligned} \tag{3.48}$$

$$\begin{aligned} &\mathbf{u}_i^{n+1} - \tilde{\mathbf{u}}_i^{n+1} + \Delta t(\nabla p_i^{n+1} + \nu\nabla\cdot\tilde{\mathbf{u}}_i^{n+1}) = 0, \\ &\nabla\cdot\mathbf{u}_i^{n+1} = 0, \mathbf{u}_i^{n+1}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad i = 1, 2. \end{aligned} \tag{3.49}$$

The above systems correspond to the projection step in the rotational pressure-correction scheme[9] for Navier–Stokes equations, and can be solved again by solving a Poisson equation with homogeneous boundary conditions.[9]

Once ϕ_i^{n+1}, μ_i^{n+1}, $\mathbf{u}_i^{n+1}$, $\tilde{\mathbf{u}}_i^{n+1}$ and p_i^{n+1} $(i = 0, 1, 2)$ are known, we can plug (3.12) into (3.43) and (3.47) to form a 2×2 linear algebraic system for ξ_1^{n+1} and ξ_2^{n+1}. We leave the detail to the interested readers.

3.3.2. *Energy stability*

The second-order scheme is also unconditionally energy stable as we show below.

Theorem 3.2. *The scheme* (3.41)–(3.47) *admits a unique solution, and is unconditionally energy stable in the sense that*

$$\tilde{E}^{n+1}(\phi, \boldsymbol{u}, r, q) - \tilde{E}^{n}(\phi, \boldsymbol{u}, r, q) \leq -\nu\Delta t\|\nabla\tilde{\boldsymbol{u}}^{n+1}\|^2 - \nu\Delta t\|\nabla\times\boldsymbol{u}^{n+1}\|^2 - \frac{2\Delta t}{T}|q^{n+1}|^2 - 2M\Delta t\|\nabla\mu^{n+1}\|^2, \tag{3.50}$$

where

$$\begin{aligned}\tilde{E}^{n+1}(\phi, \boldsymbol{u}, r, q) &= \frac{1}{2}\|\boldsymbol{u}^{n+1}\|^2 + \frac{1}{2}\|2\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\|^2 + \frac{2}{3}(\Delta t)^2\|\nabla H^{n+1}\|^2 \\ &\quad + \nu^{-1}\Delta t\|g^{n+1}\|^2 + \frac{\lambda}{2}\|\nabla\phi^{n+1}\|^2 + \frac{\lambda}{2}\|2\nabla\phi^{n+1} - \nabla\phi^n\|^2 + \frac{\lambda\gamma}{2}\|\phi^{n+1}\|^2 \\ &\quad + \frac{\lambda\gamma}{2}\|2\phi^{n+1} - \phi^n\|^2 + \lambda|r^{n+1}|^2 + \lambda|2r^{n+1} - r^n|^2 \\ &\quad + \frac{1}{2}|q^{n+1}|^2 + \frac{1}{2}|2q^{n+1} - q^n|^2,\end{aligned} \tag{3.51}$$

where $\{g^k, H^k\}$ *are defined by*

$$g^0 = 0, \quad g^{n+1} = \nu\nabla\cdot\tilde{\boldsymbol{u}}^{n+1} + g^n, \quad H^{n+1} = p^{n+1} + g^{n+1}, \quad n \geq 0. \tag{3.52}$$

Proof. By using exactly the same procedure as for the first-order scheme (3.4)–(3.8), we can show that, the second-order scheme (3.41)–(3.47) admits a unique solution.

Taking the inner products of (3.41) with $2\Delta t\mu^{n+1}$, (3.42) with $(3\phi^{n+1} - 4\phi^n + \phi^{n-1})$, respectively, and multiplying (3.43) with $4\lambda\Delta t r^{n+1}$, and using the identity

$$2(3a - 4b + c, a) = |a|^2 + |2a - b|^2 - |b|^2 - |2b - c|^2 + |a - 2b + c|^2, \tag{3.53}$$

we can obtain

$$\begin{aligned}&\frac{\lambda}{2}(\|\nabla\phi^{n+1}\|^2 + \|2\nabla\phi^{n+1} - \nabla\phi^n\|^2 - \|\nabla\phi^n\|^2 - \|2\nabla\phi^n - \nabla\phi^{n-1}\|^2) \\ &\quad + \frac{\lambda\gamma}{2}(\|\phi^{n+1}\|^2 + \|2\phi^{n+1} - \phi^n\|^2 - \|\phi^n\|^2 - \|2\phi^n - \phi^{n-1}\|^2) \\ &\quad + \lambda|r^{n+1}|^2 + \lambda|2r^{n+1} - r^n|^2 - \lambda|r^n|^2 - \lambda|2r^n - r^{n-1}|^2 \\ &\quad + \frac{\lambda}{2}\|\nabla\phi^{n+1} - 2\nabla\phi^n + \nabla\phi^{n-1}\|^2 \\ &\quad + \frac{\lambda\gamma}{2}\|\phi^{n+1} - 2\phi^n + \phi^{n-1}\|^2 + \lambda|r^{n+1} - 2r^n + r^{n-1}|^2 \\ &= -2M\Delta t\|\nabla\mu^{n+1}\|^2 - \frac{2\Delta t r^{n+1}}{\sqrt{E_1(\bar{\phi}^{n+1}) + \delta}}(\tilde{\mathbf{u}}^{n+1}, \bar{\mu}^{n+1}\nabla\bar{\phi}^{n+1}).\end{aligned} \tag{3.54}$$

Taking the inner product (3.44) with $2\Delta t\tilde{\mathbf{u}}^{n+1}$ leads to

$$\begin{aligned}&(3\tilde{\mathbf{u}}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}, \tilde{\mathbf{u}}^{n+1}) + 2\nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2\\ &\quad = -2\Delta t \exp\left(\frac{t^{n+1}}{T}\right) q^{n+1}(\bar{\mathbf{u}}^{n+1}\cdot\nabla\bar{\mathbf{u}}^{n+1}, \tilde{\mathbf{u}}^{n+1}) - 2\Delta t(\nabla p^n, \tilde{\mathbf{u}}^{n+1})\\ &\qquad + \frac{2\Delta t r^{n+1}}{\sqrt{E_1(\bar{\phi}^{n+1})+\delta}}(\bar{\mu}^{n+1}\nabla\bar{\phi}^{n+1}, \tilde{\mathbf{u}}^{n+1}).\end{aligned} \tag{3.55}$$

Recalling (3.45) and (3.53), the first term on the left-hand side of (3.55) can be transformed into

$$\begin{aligned}&(3\tilde{\mathbf{u}}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}, \tilde{\mathbf{u}}^{n+1}) = \big(3(\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^{n+1}) + 3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}, \tilde{\mathbf{u}}^{n+1}\big)\\ &\quad = 3(\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^{n+1}, \tilde{\mathbf{u}}^{n+1}) + (3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}, \mathbf{u}^{n+1})\\ &\qquad + (3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}, \tilde{\mathbf{u}}^{n+1} - \mathbf{u}^{n+1})\\ &\quad = \frac{3}{2}(\|\tilde{\mathbf{u}}^{n+1}\|^2 - \|\mathbf{u}^{n+1}\|^2 + \|\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^{n+1}\|^2) + \frac{1}{2}\|\mathbf{u}^{n+1}\|^2 + \frac{1}{2}\|2\mathbf{u}^{n+1} - \mathbf{u}^n\|^2\\ &\qquad - \frac{1}{2}\|\mathbf{u}^n\|^2 - \frac{1}{2}\|2\mathbf{u}^n - \mathbf{u}^{n-1}\|^2 + \frac{1}{2}\|\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\|^2.\end{aligned} \tag{3.56}$$

Thanks to (3.52), we can recast (3.45) as

$$\sqrt{3}\mathbf{u}^{n+1} + \frac{2}{\sqrt{3}}\Delta t\nabla H^{n+1} = \sqrt{3}\tilde{\mathbf{u}}^{n+1} + \frac{2}{\sqrt{3}}\Delta t\nabla H^n. \tag{3.57}$$

Taking the inner product of (3.57) with itself on both sides, we have

$$\begin{aligned}&3\|\mathbf{u}^{n+1}\|^2 + \frac{4}{3}(\Delta t)^2\|\nabla H^{n+1}\|^2\\ &\quad = 3\|\tilde{\mathbf{u}}^{n+1}\|^2 + \frac{4}{3}(\Delta t)^2\|\nabla H^n\|^2 + 4\Delta t(\tilde{\mathbf{u}}^{n+1}, \nabla p^n) + 4\Delta t(\tilde{\mathbf{u}}^{n+1}, \nabla g^n).\end{aligned}$$

The last term on the right-hand side can be controlled by

$$\begin{aligned}4\Delta t(\tilde{\mathbf{u}}^{n+1}, \nabla g^n) &= -4\nu^{-1}\Delta t(g^{n+1} - g^n, g^n)\\ &= 2\nu^{-1}\Delta t(\|g^n\|^2 - \|g^{n+1}\|^2 + \|g^{n+1} - g^n\|^2)\\ &= 2\nu^{-1}\Delta t\|g^n\|^2 - 2\nu^{-1}\Delta t\|g^{n+1}\|^2 + 2\nu\Delta t\|\nabla\cdot\tilde{\mathbf{u}}^{n+1}\|^2.\end{aligned} \tag{3.58}$$

Thanks to the identity

$$\|\nabla\times\mathbf{v}\|^2 + \|\nabla\cdot\mathbf{v}\|^2 = \|\nabla\mathbf{v}\|^2, \quad \forall\,\mathbf{v}\in\mathbf{H}_0^1(\Omega), \tag{3.59}$$

we have

$$\begin{aligned}4\Delta t(\tilde{\mathbf{u}}^{n+1}, \nabla g^n) &= 2\nu^{-1}\Delta t\|g^n\|^2 - 2\nu^{-1}\Delta t\|g^{n+1}\|^2\\ &\quad + 2\nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2 - 2\nu\Delta t\|\nabla\times\mathbf{u}^{n+1}\|^2.\end{aligned} \tag{3.60}$$

Then combining (3.55) with (3.56)–(3.60) results in

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}^{n+1}\|^2 + \frac{1}{2}\|2\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 + \frac{2}{3}(\Delta t)^2\|\nabla H^{n+1}\|^2 + \nu^{-1}\Delta t\|g^{n+1}\|^2 \\
&\quad + \frac{3}{2}\|\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^{n+1}\|^2 + \nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2 + \nu\Delta t\|\nabla\times\mathbf{u}^{n+1}\|^2 \\
&\le \frac{1}{2}\|\mathbf{u}^n\|^2 + \frac{1}{2}\|2\mathbf{u}^n - \mathbf{u}^{n-1}\|^2 + \frac{2}{3}(\Delta t)^2\|\nabla H^n\|^2 + \nu^{-1}\Delta t\|g^n\|^2 \\
&\quad - 2\Delta t\exp\left(\frac{t^{n+1}}{T}\right)q^{n+1}(\bar{\mathbf{u}}^{n+1}\cdot\nabla\bar{\mathbf{u}}^{n+1}, \tilde{\mathbf{u}}^{n+1}) \\
&\quad + \frac{2\Delta t r^{n+1}}{\sqrt{E_1(\bar{\phi}^{n+1}) + \delta}}(\bar{\mu}^{n+1}\nabla\bar{\phi}^{n+1}, \tilde{\mathbf{u}}^{n+1}).
\end{aligned} \tag{3.61}$$

Multiplying (3.47) by $2\Delta t q^{n+1}$ and using (3.53), we have

$$\begin{aligned}
&\frac{1}{2}|q^{n+1}|^2 + \frac{1}{2}|2q^{n+1} - q^n|^2 - \frac{1}{2}|q^n|^2 - \frac{1}{2}|2q^n - q^{n-1}|^2 + \frac{1}{2}|q^{n+1} - 2q^n + q^{n-1}|^2 \\
&= -\frac{2\Delta t}{T}|q^{n+1}|^2 + 2\Delta t\exp\left(\frac{t^{n+1}}{T}\right)q^{n+1}(\bar{\mathbf{u}}^{n+1}\cdot\nabla\bar{\mathbf{u}}^{n+1}, \tilde{\mathbf{u}}^{n+1}).
\end{aligned} \tag{3.62}$$

Then combining (3.54) with (3.61) and (3.62) leads to

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}^{n+1}\|^2 + \frac{1}{2}\|2\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 + \frac{2}{3}(\Delta t)^2\|\nabla H^{n+1}\|^2 + \nu^{-1}\Delta t\|g^{n+1}\|^2 \\
&\quad + \frac{\lambda}{2}\|\nabla\phi^{n+1}\|^2 + \frac{\lambda}{2}\|2\nabla\phi^{n+1} - \nabla\phi^n\|^2 + \frac{\lambda\gamma}{2}\|\phi^{n+1}\|^2 \\
&\quad + \frac{\lambda\gamma}{2}\|2\phi^{n+1} - \phi^n\|^2 + \lambda|r^{n+1}|^2 + \lambda|2r^{n+1} - r^n|^2 + \frac{1}{2}|q^{n+1}|^2 \\
&\quad + \frac{1}{2}|2q^{n+1} - q^n|^2 + \frac{\lambda}{2}\|\nabla\phi^{n+1} - 2\nabla\phi^n + \nabla\phi^{n-1}\|^2 \\
&\quad + \frac{\lambda\gamma}{2}\|\phi^{n+1} - 2\phi^n + \phi^{n-1}\|^2 + \frac{3}{2}\|\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^{n+1}\|^2 + \nu\Delta t\|\nabla\tilde{\mathbf{u}}^{n+1}\|^2 \\
&\quad + \nu\Delta t\|\nabla\times\mathbf{u}^{n+1}\|^2 + 2M\Delta t\|\nabla\mu^{n+1}\|^2 \\
&\quad + \lambda|r^{n+1} - 2r^n + r^{n-1}|^2 + \frac{1}{2}|q^{n+1} - 2q^n + q^{n-1}|^2 + \frac{2\Delta t}{T}|q^{n+1}|^2 \\
&= \frac{1}{2}\|\mathbf{u}^n\|^2 + \frac{1}{2}\|2\mathbf{u}^n - \mathbf{u}^{n-1}\|^2 + \frac{2}{3}(\Delta t)^2\|\nabla H^n\|^2 + \nu^{-1}\Delta t\|g^n\|^2 \\
&\quad + \frac{\lambda}{2}\|\nabla\phi^n\|^2 + \frac{\lambda}{2}\|2\nabla\phi^n - \nabla\phi^{n-1}\|^2 + \frac{\lambda\gamma}{2}\|\phi^n\|^2 + \frac{\lambda\gamma}{2}\|2\phi^n - \phi^{n-1}\|^2 \\
&\quad + \lambda|r^n|^2 + \lambda|2r^n - r^{n-1}|^2 + \frac{1}{2}|q^n|^2 + \frac{1}{2}|2q^n - q^{n-1}|^2,
\end{aligned}$$

which leads to the desired result (3.50). □

4. Error Estimates

In this section, we carry out an error analysis for the first-order semi-discrete scheme (3.4)–(3.10). While in principle the error analysis for the second-order scheme (3.41)-(3.47) can be carried out by combing the procedures below and those in Ref. 10 for the rotational pressure-correction scheme, but it will be much more involved and beyond the scope of this paper.

Since the scheme (3.4)–(3.10) is totally decoupled, it is much more difficult to carry out an error analysis as we have to deal with additional splitting errors due to the decoupling of pressure from the velocity as well as additional errors due to the explicit treatment of all nonlinear terms. Thus, extra regularity assumptions are required to carry out the error analysis.

However, the scheme avoids an essential difficulty associated with the nonlinear algebraic equation for the SAV in Refs. 17 and 34.

For notational simplicity, we shall drop the dependence on x for all functions when there is no confusion. Let $(\phi, \mu, \mathbf{u}, p, r, q)$ be the exact solution of (3.1), and $(\phi^{n+1}, \mu^{n+1}, \tilde{\mathbf{u}}^{n+1}, \mathbf{u}^{n+1}, p^{n+1}, r^{n+1}, q^{n+1})$ be the solution of the scheme (3.4)–(3.10), we denote

$$\begin{cases} \tilde{e}_{\mathbf{u}}^{n+1} = \tilde{\mathbf{u}}^{n+1} - \mathbf{u}(t^{n+1}), \quad e_{\mathbf{u}}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}(t^{n+1}), \\ e_p^{n+1} = p^{n+1} - p(t^{n+1}), \quad e_q^{n+1} = q^{n+1} - q(t^{n+1}), \\ e_\phi^{n+1} = \phi^{n+1} - \phi(t^{n+1}), \quad e_\mu^{n+1} = \mu^{n+1} - \mu(t^{n+1}), \\ e_r^{n+1} = r^{n+1} - r(t^{n+1}). \end{cases} \tag{4.1}$$

The main results are stated in the following theorem:

Theorem 4.1. *Assuming* $\phi \in W^{2,\infty}(0,T; L^2(\Omega)) \cap W^{1,\infty}(0,T; H^1(\Omega))$, $\mu \in W^{1,\infty}(0,T; H^1(\Omega)) \cap L^\infty(0,T; H^2(\Omega))$, $\boldsymbol{u} \in W^{2,\infty}(0,T; \boldsymbol{H}^{-1}(\Omega)) \cap W^{1,\infty}(0,T; \boldsymbol{H}^2(\Omega))$, *and* $p \in W^{1,\infty}(0,T; H^1(\Omega))$, *then for the first-order scheme* (3.4)–(3.10), *we have*

$$\begin{aligned} &\|\nabla e_\phi^{m+1}\|^2 + \|e_\phi^{m+1}\|^2 + \Delta t \sum_{n=0}^{m} \|\nabla e_\mu^{n+1}\|^2 + \Delta t \sum_{n=0}^{m} \|e_\mu^{n+1}\|^2 + |e_r^{m+1}|^2 \\ &\quad + \|e_{\boldsymbol{u}}^{m+1}\|^2 + \nu \Delta t \sum_{n=0}^{m} \|\nabla \tilde{e}_{\boldsymbol{u}}^{n+1}\|^2 + \Delta t \|\nabla e_p^{m+1}\|^2 + |e_q^{m+1}|^2 \\ &\quad + \sum_{n=0}^{m} \|\nabla e_\phi^{n+1} - \nabla e_\phi^n\|^2 + \sum_{n=0}^{m} \|e_\phi^{n+1} - e_\phi^n\|^2 + \sum_{n=0}^{m} \|\tilde{e}_{\boldsymbol{u}}^{n+1} - e_{\boldsymbol{u}}^n\|^2 \\ &\quad + \sum_{n=0}^{m} |e_r^{n+1} - e_r^n|^2 + \sum_{n=0}^{m} |e_q^{n+1} - e_q^n|^2 \le C(\Delta t)^2, \quad \forall\, 0 \le n \le N-1, \end{aligned}$$

where the positive constant C *is independent of* Δt.

Remark 4.1. The above result indicates that the errors for $(\phi, \mu, \mathbf{u}, r, q)$ are first-order accurate in various norms. However, it only leads to a 1/2-order error estimate for the pressure in $L^\infty(0,T;H^1)$. With the error estimates established for the velocity and phase variables, we can also derive a first-order error estimate for the pressure using a rather standard procedure, albeit very technical. For the sake of brevity, we omit the proof here and provide the detailed proof in the arXiv version of this paper.[19]

The proof of the above theorem will be carried out through a sequence of intermediate lemmas.

We shall first derive an $H^2(\Omega)$ bound for ϕ^n without assuming the Lipschitz condition on $F(\phi)$. A key ingredient is the following stability result:

$$\begin{aligned}\|\mathbf{u}^{n+1}\|^2 + \nu\Delta t \sum_{k=0}^{n} \|\tilde{\mathbf{u}}^{k+1}\|_1^2 + \Delta t \sum_{k=0}^{n} \|\nabla \mu^{k+1}\|^2 \\ + \|\phi^{n+1}\|_{H^1}^2 + |r^{n+1}|^2 + |q^{n+1}|^2 \le K_1,\end{aligned} \tag{4.2}$$

where the positive constant K_1 is dependent on $\mathbf{u}^0$ and ϕ^0, which can be derived from the unconditionally energy stability (3.33).

Lemma 4.1. *There exists a positive constant K_2 independent of Δt such that*

$$\|\Delta\phi^{n+1}\|^2 + \|\mu^{n+1}\|^2 \le K_2, \quad \forall\, 0 \le n \le N-1.$$

Proof. Combining (3.4) with (3.5) and taking the inner product with $\Delta^2\phi^{n+1}$ leads to

$$\begin{aligned}&\frac{1}{2\Delta t}(\|\Delta\phi^{n+1}\|^2 - \|\Delta\phi^n\|^2 + \|\Delta\phi^{n+1} - \Delta\phi^n\|^2) + M\lambda\|\Delta^2\phi^{n+1}\|^2 \\ &\quad + M\lambda\gamma\|\nabla\Delta\phi^{n+1}\|^2 \\ &\quad = M\lambda\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}(\Delta F'(\phi^n), \Delta^2\phi^{n+1}) - \frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}(\mathbf{u}^n \cdot \nabla\phi^n, \Delta^2\phi^{n+1}).\end{aligned} \tag{4.3}$$

Similar to the estimate in Lemma 2.4 of Ref. 24, the first term on the right-hand side of (4.3) can be controlled by the following equation with the aid of (4.2):

$$\begin{aligned}M\lambda\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}(\Delta F'(\phi^n), \Delta^2\phi^{n+1}) &\le \frac{M\lambda}{4}\|\Delta^2\phi^{n+1}\|^2 + C(K_1)\|\Delta F'(\phi^n)\|^2 \\ &\le \frac{M\lambda}{4}\|\Delta^2\phi^{n+1}\|^2 + \frac{M\lambda}{2}\|\Delta^2\phi^n\|^2 + C(K_1).\end{aligned} \tag{4.4}$$

Using (4.2) and the following Sobolev inequality:

$$\|f\|_{L^4} \le C\|f\|_{L^2}^{1/2}\|f\|_{H^1}^{1/2}, \tag{4.5}$$

the last term on the right-hand side of (4.3) can be bounded by

$$\begin{aligned}
-\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}(\mathbf{u}^n\cdot\nabla\phi^n,\Delta^2\phi^{n+1}) &\le C\|\mathbf{u}^n\|_{L^4}\|\nabla\phi^n\|_{L^4}\|\Delta^2\phi^{n+1}\| \\
&\le C\|\mathbf{u}^n\|^{1/2}\|\mathbf{u}^n\|_{H^1}^{1/2}\|\nabla\phi^n\|^{1/2}\|\nabla\phi^n\|_{H^1}^{1/2}\|\Delta^2\phi^{n+1}\| \\
&\le C\|\mathbf{u}^n\|_{H^1}\|\nabla\phi^n\|_{H^1}+\frac{M\lambda}{16}\|\Delta^2\phi^{n+1}\|^2 \\
&\le \frac{M\lambda}{4}\|\Delta^2\phi^{n+1}\|^2+C\|\mathbf{u}^n\|_{H^1}^2(\|\Delta\phi^n\|^2+C(K_1)). \qquad (4.6)
\end{aligned}$$

Combining (4.3) with (4.4)–(4.6) leads to

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\Delta\phi^{n+1}\|^2-\|\Delta\phi^n\|^2+\|\Delta\phi^{n+1}-\Delta\phi^n\|^2) \\
&\quad+\frac{M\lambda}{2}\|\Delta^2\phi^{n+1}\|^2+M\lambda\gamma\|\nabla\Delta\phi^{n+1}\|^2 \\
&\le \frac{M\lambda}{2}\|\Delta^2\phi^n\|^2+C\|\mathbf{u}^n\|_{H^1}^2(\|\Delta\phi^n\|^2+C(K_1))+C(K_1). \qquad (4.7)
\end{aligned}$$

Then multiplying (4.7) by $2\Delta t$ and summing over n, $n=0,1,2,\ldots,m$, $m\le N-1$, we have

$$\begin{aligned}
&\|\Delta\phi^{m+1}\|^2+M\lambda\Delta t\|\Delta^2\phi^{m+1}\|^2+M\lambda\gamma\Delta t\sum_{n=0}^{m}\|\nabla\Delta\phi^{n+1}\|^2 \\
&\le \|\Delta\phi^0\|^2+M\lambda\Delta t\|\Delta^2\phi^0\|^2+C\Delta t\sum_{n=0}^{m}\|\mathbf{u}^n\|_{H^1}^2\|\Delta\phi^n\|^2+C(K_1), \qquad (4.8)
\end{aligned}$$

which, together with Lemma 2.1 and Eqs. (3.5) and (4.2), lead to the desired result. □

Lemma 4.2. *Under the assumption of Theorem* 4.1, *we have*

$$\begin{aligned}
&\frac{\lambda}{2\Delta t}(\|\nabla e_\phi^{n+1}\|^2-\|\nabla e_\phi^n\|^2+\|\nabla e_\phi^{n+1}-\nabla e_\phi^n\|^2)+\frac{M}{4}\|\nabla e_\mu^{n+1}\|^2 \\
&\quad+\frac{\lambda\gamma+1}{2\Delta t}(\|e_\phi^{n+1}\|^2-\|e_\phi^n\|^2+\|e_\phi^{n+1}-e_\phi^n\|^2)+\frac{M}{4}\|e_\mu^{n+1}\|^2 \\
&\quad+\frac{\lambda}{\Delta t}(|e_r^{n+1}|^2-|e_r^n|^2+|e_r^{n+1}-e_r^n|^2) \\
&\le C\|e_\phi^n\|^2+C\|\nabla e_\phi^n\|^2+C\|\nabla e_\phi^{n+1}\|^2+C\|e_{\boldsymbol{u}}^n\|^2+C\|e_{\boldsymbol{u}}^{n+1}\|^2 \\
&\quad+C(\Delta t)^2\|\nabla(e_p^{n+1}-e_p^n)\|^2+\left(C+\frac{1}{4K_1}\|\nabla\mu^n\|^2\right)|e_r^{n+1}|^2 \\
&\quad+\frac{M}{8}\|e_\mu^n\|^2+\frac{M}{8}\|\nabla e_\mu^n\|^2+C(\Delta t)^2, \quad \forall\, 0\le n\le N-1,
\end{aligned}$$

where C is a positive constant independent of Δt.

Proof. Let R_ϕ^{n+1} be the truncation error defined by

$$R_\phi^{n+1} = \frac{\partial \phi(t^{n+1})}{\partial t} - \frac{\phi(t^{n+1}) - \phi(t^n)}{\Delta t} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (t^n - t)\frac{\partial^2 \phi}{\partial t^2} dt, \tag{4.9}$$

and E_N^{n+1} is defined by

$$E_N^{n+1} = \frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1})) + \delta}}(\mathbf{u}(t^{n+1})\cdot\nabla)\phi(t^{n+1}) - \frac{r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}(\mathbf{u}^n\cdot\nabla)\phi^n. \tag{4.10}$$

Subtracting (3.2a) at t^{n+1} from (3.4), we obtain

$$\frac{e_\phi^{n+1} - e_\phi^n}{\Delta t} - M\Delta e_\mu^{n+1} = R_\phi^{n+1} + E_N^{n+1}. \tag{4.11}$$

Taking the inner product of (4.11) with e_μ^{n+1} and e_ϕ^{n+1}, respectively, we obtain

$$\left(\frac{e_\phi^{n+1} - e_\phi^n}{\Delta t}, e_\mu^{n+1}\right) + M\|\nabla e_\mu^{n+1}\|^2 = (R_\phi^{n+1}, e_\mu^{n+1}) + (E_N^{n+1}, e_\mu^{n+1}), \tag{4.12}$$

and

$$\begin{aligned}&\frac{1}{2\Delta t}(\|e_\phi^{n+1}\|^2 - \|e_\phi^n\|^2 + \|e_\phi^{n+1} - e_\phi^n\|^2)\\ &\quad = (R_\phi^{n+1}, e_\phi^{n+1}) + (E_N^{n+1}, e_\phi^{n+1}) - M(\nabla e_\mu^{n+1}, \nabla e_\phi^{n+1}).\end{aligned} \tag{4.13}$$

Let E_F^{n+1} be defined by

$$E_F^{n+1} = \lambda r(t^{n+1})\left(\frac{F'(\phi^n)}{\sqrt{E_1(\phi^n) + \delta}} - \frac{F'(\phi(t^{n+1}))}{\sqrt{E_1(\phi(t^{n+1})) + \delta}}\right). \tag{4.14}$$

Subtracting (3.2b) at t^{n+1} from (3.5), we obtain

$$e_\mu^{n+1} = -\lambda\Delta e_\phi^{n+1} + \lambda\gamma e_\phi^{n+1} + \frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}F'(\phi^n) + E_F^{n+1}. \tag{4.15}$$

Taking the inner product of (4.15) with Me_μ^{n+1} and $\frac{e_\phi^{n+1} - e_\phi^n}{\Delta t}$, respectively, we obtain

$$\begin{aligned}M\|e_\mu^{n+1}\|^2 &= M\lambda(\nabla e_\mu^{n+1}, \nabla e_\phi^{n+1}) + M\lambda\gamma(e_\phi^{n+1}, e_\mu^{n+1})\\ &\quad + M\lambda\frac{e_r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}(F'(\phi^n), e_\mu^{n+1}) + M\left(E_F^{n+1}, e_\mu^{n+1}\right),\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}\left(\frac{e_\phi^{n+1} - e_\phi^n}{\Delta t}, e_\mu^{n+1}\right) &= \frac{\lambda}{2\Delta t}(\|\nabla e_\phi^{n+1}\|^2 - \|\nabla e_\phi^n\|^2 + \|\nabla e_\phi^{n+1} - \nabla e_\phi^n\|^2)\\ &\quad + \frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}\left(F'(\phi^n), \frac{e_\phi^{n+1} - e_\phi^n}{\Delta t}\right) + \left(E_F^{n+1}, \frac{e_\phi^{n+1} - e_\phi^n}{\Delta t}\right)\\ &\quad + \frac{\lambda\gamma}{2\Delta t}(\|e_\phi^{n+1}\|^2 - \|e_\phi^n\|^2 + \|e_\phi^{n+1} - e_\phi^n\|^2).\end{aligned} \tag{4.17}$$

Combining (4.12) with (4.13)–(4.17), we have

$$\begin{aligned}
&\frac{\lambda}{2\Delta t}(\|\nabla e_\phi^{n+1}\|^2 - \|\nabla e_\phi^n\|^2 + \|\nabla e_\phi^{n+1} - \nabla e_\phi^n\|^2) + M\|\nabla e_\mu^{n+1}\|^2 \\
&\quad + \frac{\lambda\gamma + 1}{2\Delta t}(\|e_\phi^{n+1}\|^2 - \|e_\phi^n\|^2 + \|e_\phi^{n+1} - e_\phi^n\|^2) + M\|e_\mu^{n+1}\|^2 \\
&\quad = -\frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\left(F'(\phi^n), \frac{e_\phi^{n+1} - e_\phi^n}{\Delta t}\right) - M\lambda\frac{e_r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}(F'(\phi^n), e_\mu^{n+1}) \\
&\qquad + \left(E_F^{n+1}, Me_\mu^{n+1} - \frac{e_\phi^{n+1} - e_\phi^n}{\Delta t}\right) + (E_N^{n+1}, e_\mu^{n+1} + e_\phi^{n+1}) \\
&\qquad + (R_\phi^{n+1}, e_\mu^{n+1}) + (R_\phi^{n+1}, e_\phi^{n+1}) + M\lambda\gamma(e_\phi^{n+1}, e_\mu^{n+1}). \qquad (4.18)
\end{aligned}$$

Using the Cauchy–Schwarz inequality, the second term on the right-hand side of (4.18) can be recast as

$$-M\lambda\frac{e_r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}(F'(\phi^n), e_\mu^{n+1}) \le C|e_r^{n+1}|^2 + \frac{M}{4}\|e_\mu^{n+1}\|^2. \qquad (4.19)$$

Recalling (4.11), the third term on the right-hand side of (4.18) can be written as

$$\begin{aligned}
&\left(E_F^{n+1}, Me_\mu^{n+1} - \frac{e_\phi^{n+1} - e_\phi^n}{\Delta t}\right) = (E_F^{n+1}, Me_\mu^{n+1} - M\Delta e_\mu^{n+1} - R_\phi^{n+1} - E_N^{n+1}) \\
&\quad = (E_F^{n+1}, Me_\mu^{n+1} - R_\phi^{n+1} - E_N^{n+1}) + M(\nabla E_F^{n+1}, \nabla e_\mu^{n+1}). \qquad (4.20)
\end{aligned}$$

We now estimate the terms on the right-hand side as follows: Since

$$\begin{aligned}
E_F^{n+1} &= r(t^{n+1})\left(\frac{F'(\phi^n)}{\sqrt{E_1(\phi^n)+\delta}} - \frac{F'(\phi(t^{n+1}))}{\sqrt{E_1(\phi(t^{n+1}))+\delta}}\right) \\
&= \frac{r(t^{n+1})F'(\phi(t^{n+1}))(E_1(\phi(t^{n+1})) - E_1(\phi^n))}{\sqrt{E_1(\phi^n)+\delta}\sqrt{E_1(\phi(t^{n+1}))+\delta}(\sqrt{E_1(\phi^n)+\delta} + \sqrt{E_1(\phi(t^{n+1}))+\delta})} \\
&\quad + \frac{r(t^{n+1})}{\sqrt{E_1(\phi^n)+\delta}}\left(F'(\phi^n) - F'(\phi(t^{n+1}))\right),
\end{aligned}$$

we obtain

$$\|E_F^{n+1}\| \le C\|e_\phi^n\| + C\|\phi\|_{W^{1,\infty}(0,T;L^2(\Omega))}\Delta t. \qquad (4.21)$$

Similarly, we have

$$\|\nabla E_F^{n+1}\| \le C\|e_\phi^n\| + C\|\nabla e_\phi^n\| + C\|\phi\|_{W^{1,\infty}(0,T;H^1(\Omega))}\Delta t. \qquad (4.22)$$

On the other hand,

$$E_N^{n+1} = \frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+\delta}}(\mathbf{u}(t^{n+1})\cdot\nabla)\phi(t^{n+1}) - \frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}(\mathbf{u}^n\cdot\nabla)\phi^n$$

$$= \frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+\delta}}\nabla\cdot(\mathbf{u}(t^{n+1})\phi(t^{n+1})) - \frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\nabla\cdot(\mathbf{u}^n\phi^n)$$

$$= \frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+\delta}}\nabla\cdot(\mathbf{u}(t^{n+1})\phi(t^{n+1}) - \mathbf{u}(t^n)\phi(t^n))$$

$$+ \left(\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+\delta}} - \frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\right)\nabla\cdot(\mathbf{u}(t^n)\phi(t^n))$$

$$- \frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\nabla\cdot(\mathbf{u}(t^n)e_\phi^n) - \frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\nabla\cdot(e_\mathbf{u}^n\phi^n). \tag{4.23}$$

Recalling Lemma 4.1 and (4.2), (4.5), and

$$\|f\|_{L^\infty} \le C\|f\|_{H^1}^{1/2}\|f\|_{H^2}^{1/2}, \tag{4.24}$$

the first term on the right-hand side of (4.20) can be bounded by

$$\begin{aligned}
&(E_F^{n+1}, Me_\mu^{n+1} - R_\phi^{n+1} - E_N^{n+1})\\
&\quad\le C\|E_F^{n+1}\|(\|e_\mu^{n+1}\| + \|R_\phi^{n+1}\|) + C\|\nabla E_F^{n+1}\||e_r^{n+1}|\\
&\qquad + C\|\nabla E_F^{n+1}\|(\|\mathbf{u}(t^n)\|_{L^4}\|e_\phi^n\|_{L^4} + \|e_\mathbf{u}^n\|\|\phi^n\|_{L^\infty})\\
&\qquad + C\|\nabla E_F^{n+1}\|(\|\phi\|_{W^{1,\infty}(0,T;L^2(\Omega))} + \|\mathbf{u}\|_{W^{1,\infty}(0,T;L^2(\Omega))})\Delta t\\
&\qquad + C\|\nabla E_F^{n+1}\|(\|\phi\|_{L^\infty(0,T;H^1(\Omega))} + \|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega))})\Delta t\\
&\quad\le C|e_r^{n+1}|^2 + C\|e_\phi^n\|^2 + C\|\nabla e_\phi^n\|^2 + \frac{M}{4}\|e_\mu^{n+1}\|^2 + C\|e_\mathbf{u}^n\|^2\\
&\qquad + C(\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))} + \|\mathbf{u}\|^2_{L^\infty(0,T;H^1(\Omega))})(\Delta t)^2\\
&\qquad + C(\|\phi\|^2_{W^{2,\infty}(0,T;L^2(\Omega))} + \|\phi\|^2_{W^{1,\infty}(0,T;H^1(\Omega))})(\Delta t)^2.
\end{aligned} \tag{4.25}$$

The second term on the right-hand side of (4.20) can be estimated by

$$\begin{aligned}
M(\nabla E_F^{n+1}, \nabla e_\mu^{n+1}) &\le C\|e_\phi^n\|^2 + C\|\nabla e_\phi^n\|^2 + \frac{M}{4}\|\nabla e_\mu^{n+1}\|^2\\
&\quad + C\|\phi\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^2.
\end{aligned} \tag{4.26}$$

Using (4.23), the fourth term on the right-hand side of (4.18) can be bounded by

$$\begin{aligned}
&(E_N^{n+1}, e_\mu^{n+1} + e_\phi^{n+1})\\
&\quad\le C\|\nabla e_\mu^{n+1} + \nabla e_\phi^{n+1}\|(|e_r^{n+1}| + \|\mathbf{u}(t^n)\|_{L^4}\|e_\phi^n\|_{L^4} + \|e_\mathbf{u}^n\|\|\phi^n\|_{L^\infty})\\
&\qquad + C\|\nabla e_\mu^{n+1} + \nabla e_\phi^{n+1}\|(\|\phi\|_{W^{1,\infty}(0,T;L^2(\Omega))} + \|\mathbf{u}\|_{W^{1,\infty}(0,T;L^2(\Omega))})\Delta t\\
&\qquad + C\|\nabla e_\mu^{n+1} + \nabla e_\phi^{n+1}\|(\|\phi\|_{L^\infty(0,T;H^1(\Omega))} + \|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega))})\Delta t\\
&\quad\le C|e_r^{n+1}|^2 + C\|e_\phi^n\|^2 + C\|\nabla e_\phi^n\|^2 + C\|\nabla e_\phi^{n+1}\|^2 + C\|e_\mathbf{u}^n\|^2 + \frac{M}{4}\|\nabla e_\mu^{n+1}\|^2
\end{aligned}$$

$$+ C(\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))} + \|\mathbf{u}\|^2_{L^\infty(0,T;H^1(\Omega))})(\Delta t)^2$$
$$+ C(\|\phi\|^2_{W^{1,\infty}(0,T;L^2(\Omega))} + \|\phi\|^2_{L^\infty(0,T;H^1(\Omega))})(\Delta t)^2. \tag{4.27}$$

Combining (4.18) with (4.19)–(4.27), we obtain

$$\begin{aligned}
&\frac{\lambda}{2\Delta t}(\|\nabla e_\phi^{n+1}\|^2 - \|\nabla e_\phi^n\|^2 + \|\nabla e_\phi^{n+1} - \nabla e_\phi^n\|^2) + \frac{M}{2}\|\nabla e_\mu^{n+1}\|^2 \\
&\quad + \frac{\lambda\gamma + 1}{2\Delta t}(\|e_\phi^{n+1}\|^2 - \|e_\phi^n\|^2 + \|e_\phi^{n+1} - e_\phi^n\|^2) + \frac{M}{2}\|e_\mu^{n+1}\|^2 \\
&\quad \le -\frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}\left(F'(\phi^n), \frac{e_\phi^{n+1} - e_\phi^n}{\Delta t}\right) + C|e_r^{n+1}|^2 \\
&\quad + C\|e_\phi^n\|^2 + C\|\nabla e_\phi^n\|^2 + C\|\nabla e_\phi^{n+1}\|^2 + C\|e_\mathbf{u}^n\|^2 \\
&\quad + C(\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))} + \|\mathbf{u}\|^2_{L^\infty(0,T;H^1(\Omega))})(\Delta t)^2 \\
&\quad + C(\|\phi\|^2_{W^{2,\infty}(0,T;L^2(\Omega))} + \|\phi\|^2_{W^{1,\infty}(0,T;H^1(\Omega))})(\Delta t)^2.
\end{aligned} \tag{4.28}$$

Next, we continue the estimate by establishing an error equation corresponding to the auxiliary variable r. Let R_r^{n+1} be the truncation error defined by

$$R_r^{n+1} = \frac{\partial r(t^{n+1})}{\partial t} - \frac{r(t^{n+1}) - r(t^n)}{\Delta t} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (t^n - t)\frac{\partial^2 r}{\partial t^2} dt. \tag{4.29}$$

Subtracting (3.2c) at t^{n+1} from (3.6) and multiplying the equation by $2\lambda e_r^{n+1}$ lead to

$$\begin{aligned}
&\frac{\lambda}{\Delta t}(|e_r^{n+1}|^2 - |e_r^n|^2 + |e_r^{n+1} - e_r^n|^2) \\
&\quad = \frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}\left(F'(\phi^n), \frac{e_\phi^{n+1} - e_\phi^n}{\Delta t}\right) - \frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}(F'(\phi^n), R_\phi^{n+1}) \\
&\quad + \frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}\left(F'(\phi^n) - F'(\phi(t^{n+1})), \frac{\partial \phi(t^{n+1})}{\partial t}\right) \\
&\quad + \left(\frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}} - \frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi(t^{n+1})) + \delta}}\right)\left(F'(\phi(t^{n+1})), \frac{\partial \phi(t^{n+1})}{\partial t}\right) \\
&\quad + \frac{e_r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}\left((\mu^{n+1}, \mathbf{u}^n \cdot \nabla \phi^n) - (\tilde{\mathbf{u}}^{n+1}, \mu^n \nabla \phi^n)\right) + 2\lambda R_r^{n+1} e_r^{n+1}.
\end{aligned} \tag{4.30}$$

The second term on the right-hand side of (4.30) can be estimated by

$$-\frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi^n) + \delta}}(F'(\phi^n), R_\phi^{n+1}) \le C|e_r^{n+1}|^2 + C\|\phi\|_{W^{2,\infty}(0,T;L^2(\Omega))}(\Delta t)^2. \tag{4.31}$$

The third and fourth terms on the right-hand side of (4.30) can be bounded by

$$\begin{aligned}
&\frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\left(F'(\phi^n)-F'(\phi(t^{n+1})),\frac{\partial\phi(t^{n+1})}{\partial t}\right)\\
&\quad+\left(\frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}-\frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi(t^{n+1}))+\delta}}\right)\left(F'(\phi(t^{n+1})),\frac{\partial\phi(t^{n+1})}{\partial t}\right)\\
&\quad\le C|e_r^{n+1}|^2+C||e_\phi^n||^2+C\|\phi\|^2_{W^{1,\infty}(0,T;L^2(\Omega))}(\Delta t)^2. \qquad (4.32)
\end{aligned}$$

By using (3.9), we have

$$\begin{aligned}
\tilde{e}_{\mathbf{u}}^{n+1}&=e_{\mathbf{u}}^{n+1}+\Delta t\nabla(p^{n+1}-p^n)\\
&=e_{\mathbf{u}}^{n+1}+\Delta t\nabla(e_p^{n+1}-e_p^n)+\Delta t\nabla(p(t^{n+1})-p(t^n)). \qquad (4.33)
\end{aligned}$$

Using the above (4.2) and Lemma 4.1, the fifth term on the right-hand side of (4.30) can be estimated by

$$\begin{aligned}
&\frac{e_r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\left((\mu^{n+1},\mathbf{u}^n\cdot\nabla\phi^n)-(\tilde{\mathbf{u}}^{n+1},\mu^n\nabla\phi^n)\right)\\
&\quad\le\frac{e_r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\left((\mu^{n+1},\mathbf{u}^n\cdot\nabla\phi^n)-(\mu^n,\mathbf{u}^n\cdot\nabla\phi^n)\right)\\
&\qquad+\frac{e_r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\left((\mathbf{u}^n,\mu^n\nabla\phi^n)-(\tilde{\mathbf{u}}^{n+1},\mu^n\nabla\phi^n)\right)\\
&\quad\le C|e_r^{n+1}|\|\mu^{n+1}-\mu^n\|_{L^4}\|\mathbf{u}^n\|\|\nabla\phi^n\|_{L^4}\\
&\qquad+C|e_r^{n+1}|\|\mathbf{u}^n-\tilde{\mathbf{u}}^{n+1}\|\|\mu^n\|_{L^4}\|\nabla\phi^n\|_{L^4}\\
&\quad\le\frac{M}{8}(\|e_\mu^n\|^2+\|e_\mu^{n+1}\|^2+\|\nabla e_\mu^n\|^2+\|\nabla e_\mu^{n+1}\|^2)+C\|e_{\mathbf{u}}^n\|^2+C\|e_{\mathbf{u}}^{n+1}\|^2\\
&\qquad+C(\Delta t)^2\|\nabla(e_p^{n+1}-e_p^n)\|^2+C|e_r^{n+1}|^2+\frac{1}{4K_1}\|\nabla\mu^n\|^2|e_r^{n+1}|^2\\
&\qquad+C\|p\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^4+C\|\mu\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^2\\
&\qquad+C\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))}(\Delta t)^2. \qquad (4.34)
\end{aligned}$$

Combining (4.30) with (4.31)–(4.34) results in

$$\begin{aligned}
&\frac{\lambda}{\Delta t}(|e_r^{n+1}|^2-|e_r^n|^2+|e_r^{n+1}-e_r^n|^2)\\
&\quad\le\frac{\lambda e_r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\left(F'(\phi^n),\frac{e_\phi^{n+1}-e_\phi^n}{\Delta t}\right)\\
&\qquad+\frac{M}{8}(\|e_\mu^n\|^2+\|e_\mu^{n+1}\|^2+\|\nabla e_\mu^n\|^2+\|\nabla e_\mu^{n+1}\|^2)+C\|e_{\mathbf{u}}^n\|^2+C\|e_{\mathbf{u}}^{n+1}\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ C(\Delta t)^2\|\nabla(e_p^{n+1} - e_p^n)\|^2 + \left(C + \frac{1}{4K_1}\|\nabla\mu^n\|^2\right)|e_r^{n+1}|^2 \\
&+ C|e_\phi^n|^2 + C\|p\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^4 + C\|\mu\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^2 \\
&+ C\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))}(\Delta t)^2 + C\|r\|^2_{W^{2,\infty}(0,T)}(\Delta t)^2 \\
&+ C\|\phi\|^2_{W^{2,\infty}(0,T;L^2(\Omega))}(\Delta t)^2.
\end{aligned} \tag{4.35}
$$

Combining the above equation with (4.28) gives

$$
\begin{aligned}
&\frac{\lambda}{2\Delta t}(\|\nabla e_\phi^{n+1}\|^2 - \|\nabla e_\phi^n\|^2 + \|\nabla e_\phi^{n+1} - \nabla e_\phi^n\|^2) + \frac{M}{4}\|\nabla e_\mu^{n+1}\|^2 \\
&\quad + \frac{\lambda\gamma + 1}{2\Delta t}(\|e_\phi^{n+1}\|^2 - \|e_\phi^n\|^2 + \|e_\phi^{n+1} - e_\phi^n\|^2) + \frac{M}{4}\|e_\mu^{n+1}\|^2 \\
&\quad + \frac{\lambda}{\Delta t}(|e_r^{n+1}|^2 - |e_r^n|^2 + |e_r^{n+1} - e_r^n|^2) \\
&\le C\|e_\phi^n\|^2 + C\|\nabla e_\phi^n\|^2 + C\|\nabla e_\phi^{n+1}\|^2 + C\|e_{\mathbf{u}}^n\|^2 + C\|e_{\mathbf{u}}^{n+1}\|^2 \\
&\quad + C(\Delta t)^2\|\nabla(e_p^{n+1} - e_p^n)\|^2 + \left(C + \frac{1}{4K_1}\|\nabla\mu^n\|^2\right)|e_r^{n+1}|^2 \\
&\quad + \frac{M}{8}\|e_\mu^n\|^2 + \frac{M}{8}\|\nabla e_\mu^n\|^2 + C\|r\|^2_{W^{2,\infty}(0,T)}(\Delta t)^2 \\
&\quad + C(\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))} + \|\mathbf{u}\|^2_{L^\infty(0,T;H^1(\Omega))})(\Delta t)^2 \\
&\quad + C(\|\phi\|^2_{W^{2,\infty}(0,T;L^2(\Omega))} + \|\phi\|^2_{W^{1,\infty}(0,T;H^1(\Omega))})(\Delta t)^2 \\
&\quad + C\|p\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^4 + C\|\mu\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^2,
\end{aligned} \tag{4.36}
$$

which implies the desired result. □

Lemma 4.3. *Under the assumption of Theorem* 4.1, *we have*

$$
\begin{aligned}
&\frac{\|e_{\boldsymbol{u}}^{n+1}\|^2 - \|e_{\boldsymbol{u}}^n\|^2}{2\Delta t} + \frac{\|\tilde{e}_{\boldsymbol{u}}^{n+1} - e_{\boldsymbol{u}}^n\|^2}{2\Delta t} + \frac{\nu}{2}\|\nabla\tilde{e}_{\boldsymbol{u}}^{n+1}\|^2 + \frac{\Delta t}{2}(\|\nabla e_p^{n+1}\|^2 - \|\nabla e_p^n\|^2) \\
&\quad \le -\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left((\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n, \tilde{e}_{\boldsymbol{u}}^{n+1}\right) + C(\|\boldsymbol{u}\|^2_{L^\infty(0,T;H^2(\Omega))} + \|e_{\boldsymbol{u}}^n\|_1^2)\|e_{\boldsymbol{u}}^n\|^2 \\
&\qquad + \frac{M}{16}\|e_\mu^n\|^2 + \frac{M}{16}\|\nabla e_\mu^n\|^2 + C\|\nabla e_\phi^n\|^2 + C\|e_\phi^n\|^2 + C|e_r^{n+1}|^2 + C\|e_{\boldsymbol{u}}^{n+1}\|^2 \\
&\qquad + C(\Delta t)^2(\|\nabla e_p^n\|^2 + \|\nabla e_p^{n+1}\|^2) + C(\Delta t)^2, \quad \forall\, 0 \le n \le N-1,
\end{aligned}
$$

where the positive constant C is independent of Δt.

Proof. Let $\mathbf{R}_{\mathbf{u}}^{n+1}$ be the truncation error defined by

$$\mathbf{R}_{\mathbf{u}}^{n+1} = \frac{\partial \mathbf{u}(t^{n+1})}{\partial t} - \frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (t^n - t)\frac{\partial^2 \mathbf{u}}{\partial t^2}dt. \tag{4.37}$$

Subtracting (3.2d) at t^{n+1} from (3.7), we obtain

$$\begin{aligned}\frac{\tilde{e}_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n}{\Delta t} - \nu\Delta\tilde{e}_{\mathbf{u}}^{n+1} &= \exp\left(\frac{t^{n+1}}{T}\right) q(t^{n+1})(\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1}) \\ &\quad - \exp\left(\frac{t^{n+1}}{T}\right) q^{n+1}\mathbf{u}^n\cdot\nabla\mathbf{u}^n - \nabla(p^n - p(t^{n+1})) \\ &\quad + \frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\mu^n\nabla\phi^n - \frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+\delta}}\mu(t^{n+1})\nabla\phi(t^{n+1}) + \mathbf{R}_{\mathbf{u}}^{n+1}.\end{aligned} \tag{4.38}$$

Taking the inner product of (4.38) with $\tilde{e}_{\mathbf{u}}^{n+1}$, we obtain

$$\begin{aligned}&\frac{\|\tilde{e}_{\mathbf{u}}^{n+1}\|^2 - \|e_{\mathbf{u}}^n\|^2}{2\Delta t} + \frac{\|\tilde{e}_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n\|^2}{2\Delta t} + \nu\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 \\ &\quad = \left(\exp\left(\frac{t^{n+1}}{T}\right) q(t^{n+1})(\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})\right. \\ &\qquad \left. - \exp\left(\frac{t^{n+1}}{T}\right) q^{n+1}\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{e}_{\mathbf{u}}^{n+1}\right) \\ &\qquad + \left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\mu^n\nabla\phi^n - \frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+\delta}}\mu(t^{n+1})\nabla\phi(t^{n+1}), \tilde{e}_{\mathbf{u}}^{n+1}\right) \\ &\qquad - (\nabla(p^n - p(t^{n+1})), \tilde{e}_{\mathbf{u}}^{n+1}) + (\mathbf{R}_{\mathbf{u}}^{n+1}, \tilde{e}_{\mathbf{u}}^{n+1}).\end{aligned} \tag{4.39}$$

By (3.9), we can obtain that

$$\frac{e_{\mathbf{u}}^{n+1} - \tilde{e}_{\mathbf{u}}^{n+1}}{\Delta t} + \nabla(p^{n+1} - p^n) = 0. \tag{4.40}$$

Taking the inner product of (4.40) with $\frac{e_{\mathbf{u}}^{n+1} + \tilde{e}_{\mathbf{u}}^{n+1}}{2}$, we derive

$$\frac{\|e_{\mathbf{u}}^{n+1}\|^2 - \|\tilde{e}_{\mathbf{u}}^{n+1}\|^2}{2\Delta t} + \frac{1}{2}\left(\nabla(p^{n+1} - p^n), \tilde{e}_{\mathbf{u}}^{n+1}\right) = 0. \tag{4.41}$$

Adding (4.39) and (4.41), we have

$$\begin{aligned}&\frac{\|e_{\mathbf{u}}^{n+1}\|^2 - \|e_{\mathbf{u}}^n\|^2}{2\Delta t} + \frac{\|\tilde{e}_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n\|^2}{2\Delta t} + \nu\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 \\ &\quad = \left(\exp\left(\frac{t^{n+1}}{T}\right) q(t^{n+1})(\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})\right. \\ &\qquad \left. - \exp\left(\frac{t^{n+1}}{T}\right) q^{n+1}\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{e}_{\mathbf{u}}^{n+1}\right)\end{aligned}$$

$$+\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\mu^n\nabla\phi^n-\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+\delta}}\mu(t^{n+1})\nabla\phi(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)$$
$$-\frac{1}{2}\left(\nabla(p^{n+1}+p^n-2p(t^{n+1})),\tilde{e}_{\mathbf{u}}^{n+1}\right)+(\mathbf{R}_{\mathbf{u}}^{n+1},\tilde{e}_{\mathbf{u}}^{n+1}). \tag{4.42}$$

For the first term on the right-hand side of (4.42), we have

$$\left(\exp\left(\frac{t^{n+1}}{T}\right)q(t^{n+1})(\mathbf{u}(t^{n+1})\cdot\nabla)\mathbf{u}(t^{n+1})-\exp\left(\frac{t^{n+1}}{T}\right)q^{n+1}\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)$$
$$=\exp\left(\frac{t^{n+1}}{T}\right)q(t^{n+1})\left((\mathbf{u}(t^{n+1})-\mathbf{u}^n)\cdot\nabla\mathbf{u}(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)$$
$$+\exp\left(\frac{t^{n+1}}{T}\right)q(t^{n+1})\left(\mathbf{u}^n\cdot\nabla(\mathbf{u}(t^{n+1})-\mathbf{u}^n),\tilde{e}_{\mathbf{u}}^{n+1}\right)$$
$$-\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}\left((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right). \tag{4.43}$$

Recalling (2.2) and (2.5), the first term on the right-hand side of (4.43) can be estimated by

$$\exp\left(\frac{t^{n+1}}{T}\right)q(t^{n+1})\left((\mathbf{u}(t^{n+1})-\mathbf{u}^n)\cdot\nabla\mathbf{u}(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)$$
$$\leq c_2(1+c_1)\|\mathbf{u}(t^{n+1})-\mathbf{u}^n\|\|\mathbf{u}(t^{n+1})\|_2\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|$$
$$\leq\frac{\nu}{8}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2+C\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}\|e_{\mathbf{u}}^n\|^2$$
$$+C\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))}(\Delta t)^2. \tag{4.44}$$

Noticing (4.2) and using Cauchy–Schwarz inequality, the second term on the right-hand side of (4.43) can be estimated by

$$\exp\left(\frac{t^{n+1}}{T}\right)q(t^{n+1})\left(\mathbf{u}^n\cdot\nabla(\mathbf{u}(t^{n+1})-\mathbf{u}^n),\tilde{e}_{\mathbf{u}}^{n+1}\right)$$
$$=\exp\left(\frac{t^{n+1}}{T}\right)q(t^{n+1})\left(\mathbf{u}^n\cdot\nabla(\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)),\tilde{e}_{\mathbf{u}}^{n+1}\right)$$
$$-\exp\left(\frac{t^{n+1}}{T}\right)q(t^{n+1})\left(e_{\mathbf{u}}^n\cdot\nabla e_{\mathbf{u}}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)$$
$$-\exp\left(\frac{t^{n+1}}{T}\right)q(t^{n+1})\left(\mathbf{u}(t^n)\cdot\nabla e_{\mathbf{u}}^n,\tilde{e}_{\mathbf{u}}^{n+1}\right)$$
$$\leq\frac{\nu}{8}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2+C(\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}+\|e_{\mathbf{u}}^n\|_1^2)\|e_{\mathbf{u}}^n\|^2$$
$$+C\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;H^2(\Omega))}(\Delta t)^2. \tag{4.45}$$

Then using (4.33) and (4.2), (4.5) and Lemma 4.1, the second term on the right-hand side of (4.42) can be bounded by

$$
\begin{aligned}
&\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\mu^n\nabla\phi^n-\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+\delta}}\mu(t^{n+1})\nabla\phi(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\quad=\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}\left(\mu^n\nabla\phi^n-\mu(t^{n+1})\nabla\phi(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\qquad+\left(\frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}-\frac{r(t^{n+1})}{\sqrt{E_1(\phi(t^{n+1}))+\delta}}\right)(\mu(t^{n+1})\nabla\phi(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1})\\
&\quad\le C\|\mu^n-\mu(t^{n+1})\|_{L^4}\|\nabla\phi^n\|_{L^4}\|\tilde{e}_{\mathbf{u}}^{n+1}\|\\
&\qquad+C\|\mu(t^{n+1})\|_{L^\infty(\Omega)}\|\nabla\phi^n-\nabla\phi(t^{n+1})\|\|\tilde{e}_{\mathbf{u}}^{n+1}\|\\
&\qquad+r(t^{n+1})\left(\frac{1}{\sqrt{E_1(\phi^n)+\delta}}-\frac{1}{\sqrt{E_1(\phi(t^{n+1}))+\delta}}\right)(\mu(t^{n+1})\nabla\phi(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1})\\
&\qquad+\frac{e_r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}(\mu(t^{n+1})\nabla\phi(t^{n+1}),\tilde{e}_{\mathbf{u}}^{n+1})\\
&\quad\le\frac{\nu}{8}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2+\frac{M}{16}\|e_\mu^n\|^2+\frac{M}{16}\|\nabla e_\mu^n\|^2+C\|\nabla e_\phi^n\|^2+C\|e_\phi^n\|^2+C|e_r^{n+1}|^2\\
&\qquad+C\|e_{\mathbf{u}}^{n+1}\|^2+C(\Delta t)^2\|\nabla(e_p^{n+1}-e_p^n)\|^2+C\|p\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^4\\
&\qquad+C(\|\mu\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}+\|\mu\|^2_{L^\infty(0,T;H^2(\Omega))})(\Delta t)^2\\
&\qquad+C\|\phi\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^2.
\end{aligned}\tag{4.46}
$$

Next, we estimate the third term on the right-hand side of (4.42). Using (4.33), we have

$$
\begin{aligned}
&-\frac{1}{2}\left(\nabla(p^{n+1}+p^n-2p(t^{n+1})),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\quad=-\frac{1}{2}\left(\nabla(e_p^{n+1}+e_p^n-p(t^{n+1})+p(t^n)),\tilde{e}_{\mathbf{u}}^{n+1}\right)\\
&\quad=-\frac{1}{2}(\nabla(e_p^{n+1}+e_p^n-p(t^{n+1})+p(t^n)),e_{\mathbf{u}}^{n+1}\\
&\qquad+\Delta t(\nabla(e_p^{n+1}-e_p^n)+\nabla(p(t^{n+1})-p(t^n))))\\
&\quad=-\frac{\Delta t}{2}(\|\nabla e_p^{n+1}\|^2-\|\nabla e_p^n\|^2)-\Delta t(\nabla(p(t^{n+1})-p(t^n)),\nabla e_p^n)\\
&\qquad+\frac{\Delta t}{2}\|\nabla(p(t^{n+1})-p(t^n))\|^2
\end{aligned}
$$

$$\leq -\frac{\Delta t}{2}(\|\nabla e_p^{n+1}\|^2 - \|\nabla e_p^n\|^2) + (\Delta t)^2\|\nabla e_p^n\|^2 + C\|p\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}\left((\Delta t)^2 + (\Delta t)^3\right). \quad (4.47)$$

For the last term on the right-hand side of (4.42), we have

$$(\mathbf{R}_{\mathbf{u}}^{n+1}, \tilde{e}_{\mathbf{u}}^{n+1}) \leq \frac{\nu}{8}\|\nabla \tilde{e}_{\mathbf{u}}^{n+1}\|^2 + C\|\mathbf{u}\|^2_{W^{2,\infty}(0,T;H^{-1}(\Omega))}(\Delta t)^2. \quad (4.48)$$

Combining (4.42) with (4.43)–(4.48), we obtain

$$\begin{aligned}
&\frac{\|e_{\mathbf{u}}^{n+1}\|^2 - \|e_{\mathbf{u}}^n\|^2}{2\Delta t} + \frac{\|\tilde{e}_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n\|^2}{2\Delta t} + \frac{\nu}{2}\|\nabla \tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \frac{\Delta t}{2}(\|\nabla e_p^{n+1}\|^2 - \|\nabla e_p^n\|^2)\\
&\quad \leq -\exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n, \tilde{e}_{\mathbf{u}}^{n+1}) + C(\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))} + \|e_{\mathbf{u}}^n\|_1^2)\|e_{\mathbf{u}}^n\|^2\\
&\qquad + \frac{M}{16}\|e_\mu^n\|^2 + \frac{M}{16}\|\nabla e_\mu^n\|^2 + C\|\nabla e_\phi^n\|^2 + C\|e_\phi^n\|^2\\
&\qquad + C|e_r^{n+1}|^2 + C\|e_{\mathbf{u}}^{n+1}\|^2 + C(\Delta t)^2(\|\nabla e_p^n\|^2 + \|\nabla e_p^{n+1}\|^2)\\
&\qquad + C\|p\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}\left((\Delta t)^2 + (\Delta t)^3 + (\Delta t)^4\right)\\
&\qquad + C\|\phi\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}(\Delta t)^2 + C(\|\mu\|^2_{W^{1,\infty}(0,T;H^1(\Omega))}\\
&\qquad + \|\mu\|^2_{L^\infty(0,T;H^2(\Omega))})(\Delta t)^2 + C(\|\mathbf{u}\|^2_{W^{2,\infty}(0,T;H^{-1}(\Omega))}\\
&\qquad + \|\mathbf{u}\|^2_{W^{1,\infty}(0,T;H^2(\Omega))})(\Delta t)^2,
\end{aligned}$$

which implies the desired result. □

Lemma 4.4. *Under the assumption of Theorem* 4.1, *we have*

$$\begin{aligned}
&\frac{|e_q^{n+1}|^2 - |e_q^n|^2}{2\Delta t} + \frac{|e_q^{n+1} - e_q^n|^2}{2\Delta t} + \frac{1}{2T}|e_q^{n+1}|^2\\
&\quad \leq \exp\left(\frac{t^{n+1}}{T}\right)e_q^{n+1}(\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n, \tilde{e}_{\boldsymbol{u}}^{n+1}) + C\|\boldsymbol{u}^n\|_1^2\|e_{\boldsymbol{u}}^n\|^2\\
&\quad + C(\Delta t)^2, \quad \forall\, 0 \leq n \leq N-1,
\end{aligned}$$

where the positive constant C is independent of Δt.

Proof. Subtracting (3.2f) from (3.10) leads to

$$\begin{aligned}
&\frac{e_q^{n+1} - e_q^n}{\Delta t} + \frac{1}{T}e_q^{n+1}\\
&\quad = \exp\left(\frac{t^{n+1}}{T}\right)\left((\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \tilde{\mathbf{u}}^{n+1}) - (\mathbf{u}(t^{n+1})\cdot\nabla\mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1}))\right) + \mathbf{R}_q^{n+1},
\end{aligned} \quad (4.49)$$

where

$$\mathbf{R}_q^{n+1} = \frac{\mathrm{dq(t^{n+1})}}{\mathrm{dt}} - \frac{q(t^{n+1}) - q(t^n)}{\Delta t} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (t^n - t)\frac{\partial^2 q}{\partial t^2} dt. \tag{4.50}$$

Multiplying both sides of (4.49) by e_q^{n+1} yields

$$\begin{aligned}
&\frac{|e_q^{n+1}|^2 - |e_q^n|^2}{2\Delta t} + \frac{|e_q^{n+1} - e_q^n|^2}{2\Delta t} + \frac{1}{T}|e_q^{n+1}|^2 \\
&\quad = \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}(\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \tilde{e}_{\mathbf{u}}^{n+1}) - \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \\
&\qquad \times \left(\mathbf{u}^n \cdot \nabla(\mathbf{u}(t^{n+1}) - \mathbf{u}^n), \mathbf{u}(t^{n+1})\right) - \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \\
&\qquad \times \left((\mathbf{u}(t^{n+1}) - \mathbf{u}^n) \cdot \nabla \mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1})\right) + \mathbf{R}_q^{n+1} e_q^{n+1},
\end{aligned} \tag{4.51}$$

Taking notice of (2.5) and (4.2), the second term on the right-hand side of (4.51) can be recast into

$$\begin{aligned}
&-\exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left(\mathbf{u}^n \cdot \nabla(\mathbf{u}(t^{n+1}) - \mathbf{u}^n), \mathbf{u}(t^{n+1})\right) \\
&\quad \le c_2 \exp(1)\|\mathbf{u}^n\|_1 \|\mathbf{u}(t^{n+1}) - \mathbf{u}^n\|_0 \|\mathbf{u}(t^{n+1})\|_2 |e_q^{n+1}| \\
&\quad \le \frac{1}{6T}|e_q^{n+1}|^2 + C\|\mathbf{u}^n\|_1^2 \|e_{\mathbf{u}}^n\|^2 + C\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))}\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}(\Delta t)^2.
\end{aligned} \tag{4.52}$$

The third term on the right-hand side of (4.51) can be estimated by

$$\begin{aligned}
&-\exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1} \left((\mathbf{u}(t^{n+1}) - \mathbf{u}^n) \cdot \nabla \mathbf{u}(t^{n+1}), \mathbf{u}(t^{n+1})\right) \\
&\quad \le c_2 \exp(1)\|\mathbf{u}(t^{n+1}) - \mathbf{u}^n\| \|\mathbf{u}(t^{n+1})\|_1 \|\mathbf{u}(t^{n+1})\|_2 |e_q^{n+1}| \\
&\quad \le C\|e_{\mathbf{u}}^n\|^2 + \frac{1}{6T}|e_q^{n+1}|^2 + C\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))}\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}(\Delta t)^2.
\end{aligned} \tag{4.53}$$

For the last term on the right-hand side of (4.51), we obtain

$$\mathbf{R}_q^{n+1} e_q^{n+1} \le \frac{1}{6T}|e_q^{n+1}|^2 + C|q|^2_{W^{2,\infty}(0,T)}(\Delta t)^2. \tag{4.54}$$

Combining (4.51) with (4.52)–(4.54) results in

$$\begin{aligned}
&\frac{|e_q^{n+1}|^2 - |e_q^n|^2}{2\Delta t} + \frac{|e_q^{n+1} - e_q^n|^2}{2\Delta t} + \frac{1}{2T}|e_q^{n+1}|^2 \\
&\quad \le \exp\left(\frac{t^{n+1}}{T}\right) e_q^{n+1}(\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \tilde{e}_{\mathbf{u}}^{n+1}) + C\|\mathbf{u}^n\|_1^2 \|e_{\mathbf{u}}^n\|^2 \\
&\qquad + C\|\mathbf{u}\|^2_{W^{1,\infty}(0,T;L^2(\Omega))}\|\mathbf{u}\|^2_{L^\infty(0,T;H^2(\Omega))}(\Delta t)^2 + C|q|^2_{W^{2,\infty}(0,T)}(\Delta t)^2,
\end{aligned}$$

which leads to the desired result. □

We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. We observe from (3.9) that $\mathbf{u}^{n+1} = P_{\mathbf{H}}\tilde{\mathbf{u}}^{n+1}$. Hence, (2.6) implies that $\|\mathbf{u}^{n+1}\|_1 \le C(\Omega)\|\tilde{\mathbf{u}}^{n+1}\|_1$. Using the above, Lemmas 4.2–4.4 leads to

$$\begin{aligned}
&\frac{\lambda}{2\Delta t}(\|\nabla e_\phi^{n+1}\|^2 - \|\nabla e_\phi^n\|^2 + \|\nabla e_\phi^{n+1} - \nabla e_\phi^n\|^2) + \frac{M}{4}\|\nabla e_\mu^{n+1}\|^2 \\
&\quad + \frac{\lambda\gamma+1}{2\Delta t}(\|e_\phi^{n+1}\|^2 - \|e_\phi^n\|^2 + \|e_\phi^{n+1} - e_\phi^n\|^2) + \frac{M}{4}\|e_\mu^{n+1}\|^2 \\
&\quad + \frac{\lambda}{\Delta t}(|e_r^{n+1}|^2 - |e_r^n|^2 + |e_r^{n+1} - e_r^n|^2) + \frac{\|e_{\mathbf{u}}^{n+1}\|^2 - \|e_{\mathbf{u}}^n\|^2}{2\Delta t} \\
&\quad + \frac{\|\tilde{e}_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n\|^2}{2\Delta t} + \frac{\nu}{2}\|\nabla \tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \frac{\Delta t}{2}(\|\nabla e_p^{n+1}\|^2 - \|\nabla e_p^n\|^2) \\
&\quad + \frac{|e_q^{n+1}|^2 - |e_q^n|^2}{2\Delta t} + \frac{|e_q^{n+1} - e_q^n|^2}{2\Delta t} + \frac{1}{2T}|e_q^{n+1}|^2 \\
&\le \left(C + \frac{1}{4K_1}\|\nabla\mu^n\|^2\right)|e_r^{n+1}|^2 + C(\|e_{\mathbf{u}}^n\|_1^2 + \|\mathbf{u}^n\|_1^2)\|e_{\mathbf{u}}^n\|^2 \\
&\quad + C\|\nabla e_\phi^{n+1}\|^2 + C\|\nabla e_\phi^n\|^2 + C\|e_\phi^n\|^2 \\
&\quad + C\|e_{\mathbf{u}}^{n+1}\|^2 + \frac{3}{16}M\|e_\mu^n\|^2 + \frac{3}{16}M\|\nabla e_\mu^n\|^2 + C|e_q^{n+1}|^2 \\
&\quad + C(\Delta t)^2(\|\nabla e_p^n\|^2 + \|\nabla e_p^{n+1}\|^2) + C(\Delta t)^2. \qquad (4.55)
\end{aligned}$$

Multiplying (4.55) by $2\Delta t$ and summing over n, $n = 0, 2, \ldots, m^*$, where m^* is the time step at which $|e_r^{m^*+1}|$ achieves its maximum value, and applying the discrete Gronwall Lemma 2.1, we can obtain

$$\begin{aligned}
&\|\nabla e_\phi^{m^*+1}\|^2 + \|e_\phi^{m^*+1}\|^2 + \Delta t\sum_{n=0}^{m^*}\|\nabla e_\mu^{n+1}\|^2 + \Delta t\sum_{n=0}^{m^*}\|e_\mu^{n+1}\|^2 + |e_r^{m^*+1}|^2 \\
&\quad + \|e_{\mathbf{u}}^{m^*+1}\|^2 + \nu\Delta t\sum_{n=0}^{m^*}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \Delta t\|\nabla e_p^{m^*+1}\|^2 + |e_q^{m^*+1}|^2 \\
&\quad + \sum_{n=0}^{m^*}\|\nabla e_\phi^{n+1} - \nabla e_\phi^n\|^2 + \sum_{n=0}^{m^*}\|e_\phi^{n+1} - e_\phi^n\|^2 + \sum_{n=0}^{m^*}\|\tilde{e}_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n\|^2 \\
&\quad + \sum_{n=0}^{m^*}|e_r^{n+1} - e_r^n|^2 + \sum_{n=0}^{m^*}|e_q^{n+1} - e_q^n|^2 \le C(\Delta t)^2, \qquad (4.56)
\end{aligned}$$

where we use the fact that

$$2\Delta t\sum_{n=0}^{m^*}\frac{1}{4K_1}\|\nabla\mu^n\|^2|e_r^{n+1}|^2 \le \frac{1}{2K_1}|e_r^{m^*+1}|^2\Delta t\sum_{n=0}^{m^*}\|\nabla\mu^n\|^2 \le \frac{1}{2}|e_r^{m^*+1}|^2,$$

which is a direct consequence of (4.2).

Since $|e_r^{m^*+1}| = \max_{0\le m\le N-1}|e_r^{m+1}|$, (4.56) also implies

$$\begin{aligned}&\|\nabla e_\phi^{m+1}\|^2 + \|e_\phi^{m+1}\|^2 + \Delta t\sum_{n=0}^{m}\|\nabla e_\mu^{n+1}\|^2 + \Delta t\sum_{n=0}^{m}\|e_\mu^{n+1}\|^2 + |e_r^{m+1}|^2\\ &\quad + \|e_{\mathbf{u}}^{m+1}\|^2 + \nu\Delta t\sum_{n=0}^{m}\|\nabla\tilde{e}_{\mathbf{u}}^{n+1}\|^2 + \Delta t\|\nabla e_p^{m+1}\|^2 + |e_q^{m+1}|^2\\ &\quad + \sum_{n=0}^{m}\|\nabla e_\phi^{n+1} - \nabla e_\phi^n\|^2 + \sum_{n=0}^{m}\|e_\phi^{n+1} - e_\phi^n\|^2 + \sum_{n=0}^{m}\|\tilde{e}_{\mathbf{u}}^{n+1} - e_{\mathbf{u}}^n\|^2\\ &\quad + \sum_{n=0}^{m}|e_r^{n+1} - e_r^n|^2 + \sum_{n=0}^{m}|e_q^{n+1} - e_q^n|^2 \le C(\Delta t)^2, \quad \forall\, 0\le m\le N-1,\end{aligned} \tag{4.57}$$

which implies the desired result in Theorem 4.1.

5. Numerical Results

We now provide some numerical experiments to verify our theoretical results. First, we rewrite the total energy in (3.2) as

$$E(\phi) = \int_\Omega \left\{\frac{1}{2}|\mathbf{u}|^2 + \frac{\lambda}{2}|\nabla\phi|^2 + \frac{\lambda\beta}{2\epsilon^2}\phi^2 + \frac{\lambda}{4\epsilon^2}(\phi^2-1-\beta)^2 - \lambda\frac{\beta^2+2\beta}{4\epsilon^2}\right\}d\mathbf{x}, \tag{5.1}$$

where β is a positive stabilization constant to be specified. To apply our first-order scheme (3.4)–(3.10) and second-order scheme (3.41)–(3.47) to the system (3.2), we drop the constant in the free energy and specify $E_1(\phi) = \frac{1}{4\epsilon^2}\int_\Omega(\phi^2-1-\beta)^2 d\mathbf{x}$, and modify (3.5) and (3.42) into

$$\mu^{n+1} = -\Delta\phi^{n+1} + \frac{\beta}{\epsilon^2}\phi^{n+1} + \frac{r^{n+1}}{\sqrt{E_1(\phi^n)+\delta}}F'(\phi^n), \tag{5.2}$$

$$\mu^{n+1} = -\Delta\phi^{n+1} + \frac{\beta}{\epsilon^2}\phi^{n+1} + \frac{r^{n+1}}{\sqrt{E_1(\bar{\phi}^{n+1})+\delta}}F'(\bar{\phi}^{n+1}). \tag{5.3}$$

Then we can obtain

$$F'(\phi) = \frac{\delta E_1}{\delta\phi} = \frac{1}{\epsilon^2}\phi(\phi^2-1-\beta). \tag{5.4}$$

Although we only discussed semi-discretization in time in the previous sections, the constructed MSAV schemes can be coupled with any compatible spatial discretization. In this section, we give the fully discrete first-order MSAV scheme based on the MAC (marker and cell) method on the staggered grids as follows: Find

$(\phi_h^{n+1}, \mu_h^{n+1}, \tilde{\mathbf{u}}_h^{n+1}, \mathbf{u}_h^{n+1}, p_h^{n+1}, r_h^{n+1}, q_h^{n+1})$ such that

$$\frac{\phi_h^{n+1} - \phi_h^n}{\Delta t} + \frac{r_h^{n+1}}{\sqrt{E_1^h(\phi_h^n) + \delta}} \mathcal{P}_h^y \mathcal{P}_h^x [\mathbf{u}_{h,1}^n D_x \phi_h^n + \mathbf{u}_{h,2}^n D_y \phi_h^n] = M[d_x D_x \mu_h^{n+1} + d_y D_y \mu_h^{n+1}], \tag{5.5}$$

$$\mu_h^{n+1} = -\lambda[d_x D_x \phi_h^{n+1} + d_y D_y \phi_h^{n+1}] + \lambda\gamma\phi_h^{n+1} + \lambda \frac{r_h^{n+1}}{\sqrt{E_1^h(\phi_h^n) + \delta}} F'(\phi_h^n); \tag{5.6}$$

$$\begin{aligned} \frac{r_h^{n+1} - r_h^n}{\Delta t} &= \frac{1}{2\sqrt{E_1^h(\phi^n) + \delta}} \left(F'(\phi_h^n), \frac{\phi_h^{n+1} - \phi_h^n}{\Delta t} \right)_{l^2,M} \\ &\quad + \frac{1}{2\lambda\sqrt{E_1^h(\phi^n) + \delta}} (\mu_h^{n+1}, \mathcal{P}_h^y \mathcal{P}_h^x [\mathbf{u}_{h,1}^n D_x \phi_h^n + \mathbf{u}_{h,2}^n D_y \phi_h^n])_{l^2,M} \\ &\quad - \frac{1}{2\lambda\sqrt{E_1^h(\phi_h^n) + \delta}} \left(\tilde{\mathbf{u}}_h^{n+1}, \mu_h^n D\phi_h^n \right)_{l^2}; \end{aligned} \tag{5.7}$$

$$\begin{aligned} &\frac{\tilde{\mathbf{u}}_{h,1}^{n+1} - \mathbf{u}_{h,1}^n}{\Delta t} + \exp\left(\frac{t^{n+1}}{T}\right) q_h^{n+1} \left(\mathbf{u}_{h,1}^n D_x(\mathcal{P}_h^x \mathbf{u}_{h,1}^n) + \mathcal{P}_h^y(\mathcal{P}_h^x \mathbf{u}_{h,2}^n D_y \mathbf{u}_{h,1}^n) \right) \\ &\quad - \nu D_x(d_x \tilde{\mathbf{u}}_{h,1})^{n+1} - \nu d_y(D_y \tilde{\mathbf{u}}_{h,1})^{n+1} + D_x p_h^n = \frac{r_h^{n+1}}{\sqrt{E_1^h(\phi_h^n) + \delta}} \mu_h^n D_x \phi_h^n, \end{aligned} \tag{5.8}$$

$$\begin{aligned} &\frac{\tilde{\mathbf{u}}_{h,2}^{n+1} - \mathbf{u}_{h,2}^n}{\Delta t} + \exp\left(\frac{t^{n+1}}{T}\right) q_h^{n+1} \left(\mathcal{P}_h^x(\mathcal{P}_h^y \mathbf{u}_{h,1}^n D_x \mathbf{u}_{h,2}^n) + \mathbf{u}_{h,2}^n D_y(\mathcal{P}_h^y \mathbf{u}_{h,2}^n) \right) \\ &\quad - \nu d_x(D_x \tilde{\mathbf{u}}_{h,2})^{n+1} - \nu D_y(d_y \tilde{\mathbf{u}}_{h,2})^{n+1} + D_y p_h^n = \frac{r_h^{n+1}}{\sqrt{E_1^h(\phi_h^n) + \delta}} \mu_h^n D_y \phi_h^n, \end{aligned} \tag{5.9}$$

$$d_x \mathbf{u}_{h,1}^{n+1} + d_y \mathbf{u}_{h,2}^{n+1} = 0, \tag{5.10}$$

$$\frac{\mathbf{u}_h^{n+1} - \tilde{\mathbf{u}}_h^{n+1}}{\Delta t} + D(p_h^{n+1} - p_h^n) = 0, \tag{5.11}$$

$$\begin{aligned} \frac{q_h^{n+1} - q_h^n}{\Delta t} &= \exp\left(\frac{t^{n+1}}{T}\right) \left(\left(\mathbf{u}_{h,1}^n D_x(\mathcal{P}_h^x \mathbf{u}_{h,1}^n) + \mathcal{P}_h^y(\mathcal{P}_h^x \mathbf{u}_{h,2}^n D_y \mathbf{u}_{h,1}^n) \right), \tilde{\mathbf{u}}_{h,1}^{n+1} \right)_{l^2,T,M} \\ &\quad + \exp\left(\frac{t^{n+1}}{T}\right) \left(\left(\mathcal{P}_h^x(\mathcal{P}_h^y \mathbf{u}_{h,1}^n D_x \mathbf{u}_{h,2}^n) + \mathbf{u}_{h,2}^n D_y(\mathcal{P}_h^y \mathbf{u}_{h,2}^n) \right), \tilde{\mathbf{u}}_{h,2}^{n+1} \right)_{l^2,M,T} - \frac{1}{T} q_h^{n+1}, \end{aligned} \tag{5.12}$$

where $\mathcal{P}_h^x$ and $\mathcal{P}_h^y$ are linear interpolation operators in the x and y directions, respectively, and we use exactly the same notations, such as the discrete difference quotient operators, the inner products, and the boundary conditions as in Ref. 18. The second-order fully discrete scheme can be obtained by using exactly similar procedure.

For simplicity, we define

$$\begin{cases} \|f-g\|_{l^\infty} = \max\limits_{0\le n\le m} \{\|f^{n+1}-g^{n+1}\|\}, \\ \|f-g\|_{l^2} = \left(\sum\limits_{n=0}^{m} \Delta t \|f^{n+1}-g^{n+1}\|^2\right)^{1/2}, \\ \|R-r\|_\infty = \max\limits_{0\le n\le m} \{R^{n+1}-r^{n+1}\}. \end{cases}$$

5.1. *Convergence tests*

In the following simulation, we choose $\Omega = (0,1)\times(0,1)$, $\beta = 5$, $T = 0.1$, $\lambda = \gamma = 1$, $\nu = 0.001$, $\epsilon = 0.3$, $M = 0.001$, $\delta = 0$, with the initial condition

$$\begin{aligned} &\mathbf{u}^0(x,y) = [\sin^2(\pi x)\sin(2\pi y),\, -\sin^2(\pi y)\sin(2\pi x)], \quad p^0(x,y) = 0; \\ &\phi^0(x,y) = \cos(\pi x)\cos(\pi y), \quad r^0 = \sqrt{E_1(\phi^0)+\delta}, \quad q^0 = 1. \end{aligned} \tag{5.13}$$

The spatial discretization is based on the MAC scheme on the staggered grid with $N_x = N_y = 160$ so that the spatial discretization error is negligible compared to the time discretization error for the time steps used in the simulation.

We measure the Cauchy error due to the fact that we do not have possession of an exact solution. Specifically, the error between two different time step sizes Δt and $\frac{\Delta t}{2}$ is calculated by $\|e_\zeta\| = \|\zeta_{\Delta t} - \zeta_{\Delta t/2}\|$. We present numerical results for the first- and second-order schemes (3.4)–(3.10) and (3.41)–(3.47) in Tables 1–4. From Tables 1 and 2, we observe that the first-order scheme leads to first-order estimates for all functions, consistent with error estimates in Theorem 4.1. We also observe from Table 4 that the second-order scheme (3.41)–(3.47) leads to second-order convergence for all functions except for pressure which converges at rate 3/2 as predicted by the error estimates in Ref. 10 for second-order rotational pressure-correction scheme for the Navier–Stokes equations.

Table 1. Errors and convergence rates with the first-order scheme (3.4)–(3.8).

Δt	$\|e_\phi\|_{l^\infty}$	Rate	$\|\nabla e_\phi\|_{l^\infty}$	Rate	$\|e_r\|_\infty$	Rate
2^{-3}	3.52E-3	—	2.52E-2	—	1.87E-3	—
2^{-4}	2.44E-3	0.53	1.63E-2	0.63	8.15E-4	1.20
2^{-5}	1.43E-3	0.77	9.41E-3	0.80	3.88E-4	1.07
2^{-6}	7.74E-4	0.89	5.06E-3	0.89	1.91E-4	1.02

Table 2. Errors and convergence rates with the first-order scheme (3.4)–(3.8).

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|\nabla \tilde{e}_{\mathbf{u}}\|_{l^2}$	Rate	$\|e_p\|_{l^2}$	Rate	$\|e_q\|_\infty$	Rate
2^{-3}	4.14E-2	—	1.35E-3	—	1.47E-2	—	1.07E-2	—
2^{-4}	2.18E-2	0.93	6.80E-4	0.99	6.34E-3	1.21	5.53E-3	0.96
2^{-5}	1.14E-2	0.94	3.57E-4	0.93	3.03E-3	1.07	2.81E-3	0.98
2^{-6}	5.83E-3	0.96	1.85E-4	0.95	1.50E-3	1.01	1.41E-3	0.99

Table 3. Errors and convergence rates with the second-order scheme (3.41)–(3.47).

Δt	$\|e_\phi\|_{l^\infty}$	Rate	$\|\nabla e_\phi\|_{l^\infty}$	Rate	$\|e_r\|_\infty$	Rate
2^{-3}	1.62E-3	—	1.12E-2	—	7.85E-4	—
2^{-4}	3.75E-4	2.11	2.57E-3	2.12	1.67E-4	2.23
2^{-5}	8.97E-5	2.06	6.11E-4	2.07	3.95E-5	2.08
2^{-6}	2.17E-5	2.05	1.48E-4	2.05	9.71E-6	2.03

Table 4. Errors and convergence rates with the second-order scheme (3.41)–(3.47).

Δt	$\|e_{\mathbf{u}}\|_{l^\infty}$	Rate	$\|\nabla \tilde{e}_{\mathbf{u}}\|_{l^2}$	Rate	$\|e_p\|_{l^2}$	Rate	$\|e_q\|_\infty$	Rate
2^{-3}	7.72E-3	—	1.20E-3	—	3.63E-2	—	2.12E-3	—
2^{-4}	1.50E-3	2.36	2.98E-4	2.01	1.28E-2	1.50	5.01E-4	2.08
2^{-5}	3.35E-4	2.17	7.97E-5	1.90	4.56E-3	1.49	1.23E-4	2.03
2^{-6}	8.33E-5	2.01	2.21E-5	1.85	1.62E-3	1.50	3.03E-5	2.01

5.2. *Coarsening dynamics*

In the following simulation, we choose $\Omega = (0,1)\times(0,1)$, $T = 5$, $\Delta t = 0.001$, $\lambda = 0.02$, $\nu = 1$, $\epsilon = 0.01$, $M = 1e{-}4$, $\delta = 10$, with the a random initial condition for the phase function with values in $[-0.1, 0.1]$. The spatial discretization is based on the MAC scheme on the staggered grid with $N_x = N_y = 100$. We present characteristic evolutions of the phase-field variable with first-order scheme at different times $t = 0.15, 0.3, 0.5, 0.8, 1, 1.5, 3, 5$, respectively, in Fig. 1. In addition, the energy decay curve of the original energy and modified energy with first-order scheme for the Cahn–Hilliard–Navier–Stokes model is shown in Fig. 2, which demonstrates the consistency between the original energy and modified energy.

As a comparison, we also implement the following usual second-order semi-implicit scheme: Find $(\phi^{n+1}, \mu^{n+1}, \tilde{\mathbf{u}}^{n+1}, \mathbf{u}^{n+1}, p^{n+1})$ such that

$$\frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\Delta t} + (\bar{\mathbf{u}}^{n+1}\cdot\nabla)\bar{\phi}^{n+1} = M\Delta\mu^{n+1}, \tag{5.14}$$

$$\mu^{n+1} = -\lambda\Delta\phi^{n+1} + \lambda\gamma\phi^{n+1} + \lambda F'(\bar{\phi}^{n+1}); \tag{5.15}$$

$$\begin{aligned}&\frac{3\tilde{\mathbf{u}}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} + \bar{\mathbf{u}}^{n+1}\cdot\nabla\bar{\mathbf{u}}^{n+1} - \nu\Delta\tilde{\mathbf{u}}^{n+1} + \nabla p^n\\ &\quad = \bar{\mu}^{n+1}\nabla\bar{\phi}^{n+1}, \tilde{\mathbf{u}}^{n+1}|_{\partial\Omega} = 0;\end{aligned} \tag{5.16}$$

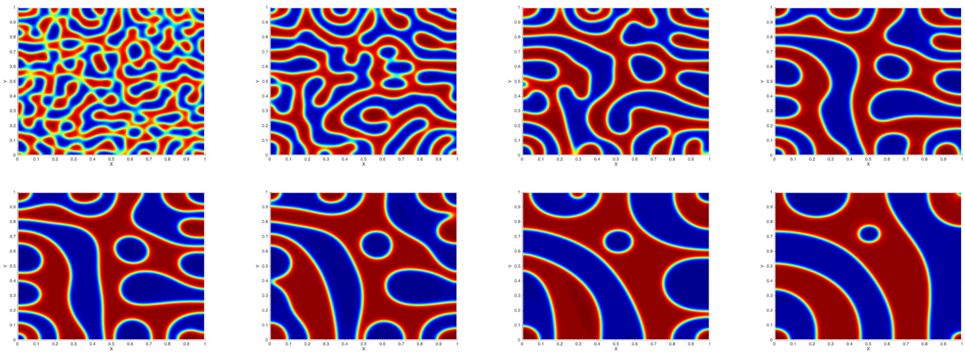

Fig. 1. Characteristic evolutions of the phase-field variable at different times with $t = 0.15, 0.3, 0.5, 0.8, 1, 1.5, 3, 5$, respectively.

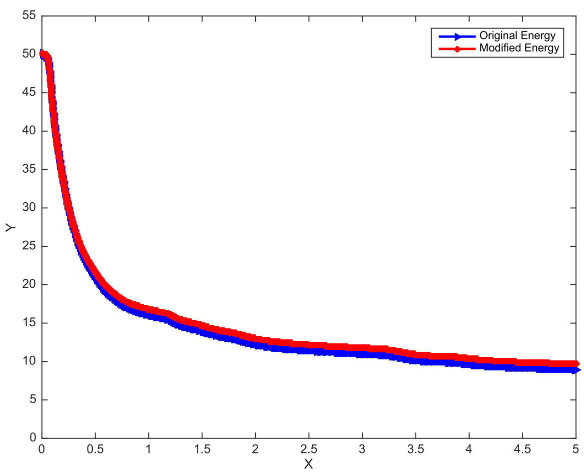

Fig. 2. Evolution of the original energy and modified energy.

$$\frac{3\mathbf{u}^{n+1} - 3\tilde{\mathbf{u}}^{n+1}}{2\Delta t} + \nabla(p^{n+1} - p^n + \nu\nabla \cdot \tilde{\mathbf{u}}^{n+1}) = 0, \tag{5.17}$$

$$\nabla \cdot \mathbf{u}^{n+1} = 0, \quad \mathbf{u}^{n+1} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{5.18}$$

where $\bar{g}^{n+1} = 2g^n - g^{n-1}$ for any sequence $\{g^n\}$.

We plot the free energy as function of time with $\Delta t = 3e - 3$ by using the semi-implicit scheme (5.14)–(5.18) and MSAV scheme (3.41)–(3.47) (see Fig. 3). We observe that the free energy by the semi-implicit scheme (5.14)–(5.18) eventually increases, violating the energy dissipation law, while the free energy by the MSAV scheme (3.41)–(3.47) remains to be dissipative.

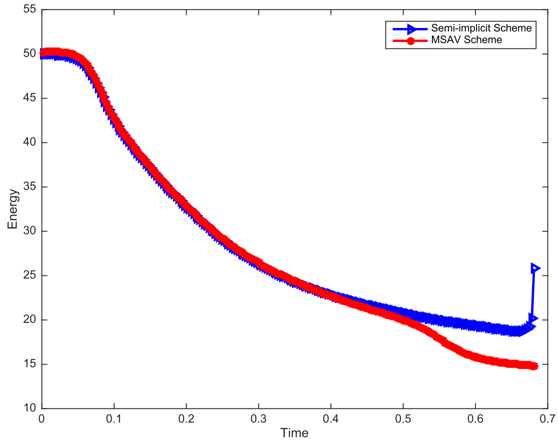

Fig. 3. Free energy curves for the semi-implicit scheme and MSAV scheme.

6. Concluding Remarks

The Cahn–Hilliard–Navier–Stokes phase field model is a highly coupled nonlinear system whose energy dissipation relies on delicate cancellations of nonlinear interactions. We constructed in this paper efficient time discretization schemes for the Cahn–Hilliard–Navier–Stokes phase field model by combining the MSAV approach to deal with the various nonlinear terms and the standard or rotational pressure-correction to deal with the coupling of pressure and velocity. The resulting first- and second-order schemes are fully decoupled, linear, unconditional energy stable and only require solving several elliptic equations with constant coefficients at each time step. So they are very efficient and easy to implement. We also carried out a rigorous error analysis for the first-order scheme and derived optimal error estimates for all relevant functions in different norms.

We only carried out error analysis for the first-order scheme. It is hopeful that second-order error estimates could be derived by combing the approach in this paper with the techniques used to derive second-order error estimates for the rotational pressure-correction scheme in Ref. 10. But this process is nontrivial and requires substantial new efforts. On the other hand, we have only considered time discretization in this work. While the stability proofs and error estimates are based on weak formulations with simple test functions, it is still a big challenge to extend this approach to fully discrete schemes with properly formulated spatial discretization. Error analysis for the second-order scheme as well for full discretization will be left as subjects of future endeavor.

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