Pattern formations of 2D Rayleigh–Bénard convection with no-slip boundary conditions for the velocity at the critical length scales

Taylan Sengul, Jie Shen and Shouhong Wang*†

Communicated by R. Showalter

We study the Rayleigh–Bénard convection in a 2D rectangular domain with no-slip boundary conditions for the velocity. The main mathematical challenge is due to the no-slip boundary conditions, because the separation of variables for the linear eigenvalue problem, which works in the free-slip case, is no longer possible. It is well known that as the Rayleigh number crosses a critical threshold $R_c$, the system bifurcates to an attractor, which is an $(m-1)$-dimensional sphere, where $m$ is the number of eigenvalues, which cross zero as $R$ crosses $R_c$. The main objective of this article is to derive a full classification of the structure of this bifurcated attractor when $m = 2$. More precisely, we rigorously prove that when $m = 2$, the bifurcated attractor is homeomorphic to a one-dimensional circle consisting of exactly four or eight steady states and their connecting heteroclinic orbits. In addition, we show that the mixed modes can be stable steady states for small Prandtl numbers. Copyright © 2014 John Wiley & Sons, Ltd.

Keywords: Bénard convection; dynamic transition; pattern formation

1. Introduction

The Rayleigh–Bénard convection problem is one of the fundamental problems in the physics of fluids. The basic phenomena of the Rayleigh–Bénard convection in horizontally extended systems are widely known. The influence of the side walls, although not studied as thoroughly as the horizontally extended case, is of practical importance for engineering applications.

In this paper, we study the Rayleigh–Bénard convection in a 2D rectangular domain with no-slip boundary conditions for the velocity. This problem is also closely related to the problem of infinite channel with rectangular cross section, which has been studied by Davies-Jones [1], Luijkx and Platten [2], and Kato and Fujimura [3], among others.

The linear aspects of the problem we consider in this paper have been studied by Lee et al. [4], Mizushima [5], and Gelfgat [6]. In these papers, the critical Rayleigh number and the structure of the critical eigenmodes have been studied for small aspect ratio containers.

From dynamical transition and pattern formation point of view, Ma and Wang [7, 8] proved that under some general boundary conditions, the problem always undergoes a dynamic transition to an attractor $\Sigma_0$ as the Rayleigh number $R$ crosses the first critical Rayleigh number $R_c$. They also proved that the bifurcated attractor is homological to $S^{m-1}$, where $m$ is the number of critical eigenmodes.

In the 2D setting that we consider, $m$ is either 1 or 2, and the latter case can only happen at the critical length scales where two modes with wave numbers $k$ and $k + 1$ become critical simultaneously. When $m = 1$, the structure of $\Sigma_0$ is trivial, which is merely a disjoint union of two attracting steady states. Thus, our task in this paper is to classify the structure of the attractor when $m = 2$. This has been studied recently in [9] for the 3D Rayleigh–Bénard problem where the boundaries were assumed to be free slip for the velocity and the wave numbers of the critical modes were assumed to be equal.

The main mathematical challenge in this paper is due to the no-slip boundary conditions because the separation of variables for the linear eigenvalue problem, which works in the free-slip case, is not possible anymore. To overcome this difficulty, the main approach for our study is to combine rigorous analysis and numerical computation using spectral method.
As we know, spectral methods have long been used to address the hydrodynamic instability problems. In fact, in his seminal work [10], Orszag studied the classical Orr–Sommerfeld linear instability problem using a Chebyshev-tau method. In this paper, to treat the linear eigenvalue problem, we employ a Legendre–Galerkin method where compact combinations of Legendre polynomials, also called generalized Jacobi polynomials, satisfying all the boundary conditions are used as trial functions. The main advantage of Legendre–Galerkin method is that the resulting matrices are sparse, which allows for a very efficient and accurate solution of the linearized problem; see also Hill and Straughan [11] and Gheorghiu and Dragomirescu [12].

Once the eigenpairs of the linear problem are identified, the transition analysis is carried out by reducing the infinite dimensional system to the center manifold in the first two critical eigendirections. The coefficients of this reduced system are calculated numerically. Our main results are described later.

We first classify the eigenmodes into four classes according to their parities using the symmetry of the problem. Then, we numerically show that the first two unstable modes are always parity class 1 or 2. Then, we study the transition near the critical length scales where two eigenvalues become positive simultaneously. Next, we rigorously prove that the local attractor at small supercritical Rayleigh numbers is in fact homeomorphic to the circle, which has four or eight steady states with half of them as stable points and the rest as saddle points. The critical eigenmodes are always bifurcated steady states on the attractor, and when the attractor has eight steady states, the mixed modes, which are superpositions of the critical eigenmodes, are also bifurcated.

Second, let \( \beta_1 \) and \( \beta_2 \) denote the two largest eigenvalues of the linearized problem. We find that a small neighborhood of \( \beta_1 = \beta_2 = 0 \) in the \( \beta_1–\beta_2 \) plane can be separated into several sectors with different asymptotical structures. In particular, we find that there is a critical Prandtl number \( Pr_c \) for the first two critical length scales \( L = 1.5702 \text{ and } L = 2.6611 \), such that for \( Pr < Pr_c \), there is a sector in this plane for which mixed modes are stable fixed points of the attractor. For \( Pr > Pr_c \), the mixed modes are never stable, and instead, there is a sector in this plane in which both of the critical eigenmodes coexist as stable steady states. In this case, the initial conditions determine which one of these eigenmodes will be realized. The critical Prandtl number \( Pr_c \) is around 0.14 for the first critical length scale \( L = 1.5702 \) and around 0.05 for the second critical length scale \( L = 2.6611 \). For higher critical length scales, we found that mixed modes are never stable points of the attractor.

Third, recently, Ma and Wang have developed the dynamic transition theory to study transition and bifurcation problems in nonlinear sciences; see [13]. This paper is a first attempt to combine this theory with the numerical tools of the spectral methods to study the detailed structure of the transition and pattern formation.

The paper is organized as follows. In Section 2, the governing equations and the functional setting of the problem are discussed. In Section 3, linear eigenvalue problem is studied. Section 4 states the main theorem. Section 5 is devoted to the proof of the main theorem. In Section 6, we demonstrate a method to compute the coefficients of the reduced system. And the last section discusses the results obtained by our analysis.

2. Governing equations and the functional setting

Two-dimensional thermal convection with no-slip, perfectly conducting boundaries can be modeled by the Boussinesq equations. The governing equations on the rectangular domain \( \Omega = (0, L) \times (0, 1) \in \mathbb{R}^2 \) read as

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u &= Pr(-\nabla p + \Delta u) + \sqrt{R\Pr} \theta \mathbf{k}, \\
\frac{\partial \theta}{\partial t} + (u \cdot \nabla) \theta &= \sqrt{R\Pr \mathbf{w}} + \Delta \theta, \\
\nabla \cdot u &= 0.
\end{align*}
\]

(1)

Here, \( u = (u, w) \) is the velocity field, \( \theta \) is the temperature field, and \( p \) is the pressure field. These fields represent a perturbation around the motionless state with a linear temperature profile. The dimensionless numbers are the Prandtl number \( Pr \) and the Rayleigh number \( R \), which is also the control parameter. \( \mathbf{k} \) represents the unit vector in the z-direction.

The equations (1) are supplemented with no-slip boundary conditions for the velocity and perfectly conducting boundary conditions for the temperature.

\[
\begin{align*}
\mathbf{u} &= \theta = 0, \quad \text{on } \partial \Omega.
\end{align*}
\]

(2)

For the functional setting, we define the relevant function spaces:

\[
\begin{align*}
H &= \{(u, \theta) \in L^2(\Omega, \mathbb{R}^3) : \nabla \cdot u = 0, \mathbf{u} \cdot n |_{\partial \Omega} = 0 \}, \\
H_1 &= \{(u, \theta) \in H^2(\Omega, \mathbb{R}^3) : \nabla \cdot u = 0, \mathbf{u} |_{\partial \Omega} = 0, \theta |_{\partial \Omega} = 0 \}.
\end{align*}
\]

(3)

For \( \phi = (u, \theta) \), let \( G : H_1 \to H \) and \( L_{\phi} : H_1 \to H \) be defined by

\[
\begin{align*}
L_{\phi} \phi &= \left( Pr \Delta u + \sqrt{R\Pr} \theta \mathbf{k}, \sqrt{R\Pr \mathbf{w}} + \Delta \theta \right), \\
G(\phi) &= -\left( \nabla (u \cdot \nabla) u, (u \cdot \nabla) \phi \right),
\end{align*}
\]

(4)

with \( \mathcal{P} \) denoting the Leray projection onto the divergence-free vectors.
The equations (1)–(2) supplemented with initial conditions can be put into the following abstract ordinary differential equation:

\[ \frac{d\phi}{dt} = L_R\phi + G(\phi), \quad \phi(0) = \phi_0. \]  

(5)

For results concerning the existence and uniqueness of solutions of (5), we refer to Foias et al. [14]. Finally, for \( \phi_i = (u_i, \theta_i) \), \( u_i = (u_i, w_i, \theta_i) \), \( i = 1, 2, 3 \), we define the following trilinear forms.

\[
\begin{align*}
G(\phi_1, \phi_2, \phi_3) &= -\int_\Omega (u_1 \cdot \nabla)u_2 \cdot u_3 \, dx \, dz - \int_\Omega (u_1 \cdot \nabla)\theta_2 \cdot \theta_3 \, dx \, dz, \\
G_i(\phi_1, \phi_2, \phi_3) &= G(\phi_1, \phi_2, \phi_3) + G(\phi_2, \phi_1, \phi_3).
\end{align*}
\]

(6)

3. Linear analysis

We first study the eigenvalue problem \( L_R\phi = \beta\phi \), which reads as

\[
\text{Pr} \left( \Delta u - \frac{\partial p}{\partial x} \right) = \beta u, \\
\text{Pr} \left( \Delta w - \frac{\partial p}{\partial z} \right) + \sqrt{\text{Pr}} \sqrt{\text{Pr}} \theta = \beta w, \\
\Delta \theta + \sqrt{\text{Pr}} \sqrt{\text{Pr}} w = \beta \theta, \\
div u = 0, \\
u = \theta = 0, \quad \text{at } \partial \Omega.
\]

(7)

In the following, we list some of the properties of this eigenvalue problem.

1. The linear operator \( L_R \) is symmetric. Hence, the eigenvalues \( \beta_i \) are real, and the eigenfunctions \( \phi_i \) are orthogonal with respect to \( L^2 \)-inner product. Moreover, there is a sequence

\[ 0 < R_1 \leq R_2 \leq \cdots \]

such that \( \beta_i(R_i) = 0 \). \( R_i \) is found by setting \( \beta = 0 \) in (7). In this case, the problem becomes an eigenvalue problem with \( \sqrt{R} \) as the eigenvalue.

2. We have

\[
\beta_i(R) \geq 0 \quad \text{if } R \leq R_i,
\]

which can be seen by computing the derivative of \( \beta_i \) with respect to \( R \) at \( R = R_i \).

\[
\frac{d\beta_i}{dR} \bigg|_{R=R_i} = \frac{1}{\text{Pr}} \int_\Omega \frac{\sqrt{\text{Pr}} \theta_i w_i \, dx \, dz}{\left( u_i^2 + w_i^2 + \theta_i^2 \right)^{3/2}} \, dx \, dz,
\]

(8)

where \((u_i, w_i, \theta_i)\) is the \( i \)-th eigenfunction. Also at \( R = R_i \) by the third equation in (7), \( w_i = -R_i^{-1/2} \text{Pr}^{-1/2} \Delta \theta_i \) as \( \beta_i(R_i) = 0 \). Plugging these into (8) and integrating by parts, we see that

\[
\frac{d\beta_i}{dR} \bigg|_{R=R_i} = \frac{1}{\text{Pr}} \int_\Omega \frac{\left| \nabla \theta_i \right|^2 \, dx \, dz}{\left( u_i^2 + w_i^2 + \left| \theta_i \right|^2 \right)^{3/2}} \, dx \, dz > 0.
\]

3. We denote the critical Rayleigh number \( R_c = R_i \). That is,

\[
\beta_i(R) \begin{cases} < 0 & \text{if } R < R_c, \\ = 0 & \text{if } R = R_c, \\ > 0 & \text{if } R > R_c \end{cases}, \quad i = 1, \ldots, m
\]

\[
\beta_i(R_c) < 0, \quad i > m.
\]

(9)

\( m \) in (9) does not depend on the Prandtl number \( \text{Pr} \) but only on \( L \). To see this, one makes the change of variable \( \theta = \sqrt{\text{Pr}} \theta' \) so that the solution of (7) for the eigenvalue of \( \beta = 0 \) is independent of \( \text{Pr} \). By simplicity of the first eigenvalue (see Theorem 3.7 in the study of Ma and Wang [15]), for almost every value of \( L \) except a discrete set of values, \( m \) in (9) is 1.
Table I. Possible parity classes of the eigenfunctions of the linear operator.

<table>
<thead>
<tr>
<th>Class 1</th>
<th>Class 2</th>
<th>Class 3</th>
<th>Class 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(e, e), \theta(o, e)$</td>
<td>$\psi(o, e), \theta(e, e)$</td>
<td>$\psi(e, o), \theta(e, o)$</td>
<td>$\psi(o, o), \theta(e, o)$</td>
</tr>
</tbody>
</table>

Introducing the stream function $\psi_z = u, \psi_x = -w$, we can eliminate the pressure $p$ from the linear eigenvalue problem (7).

$$\begin{align*}
\text{Pr} \Delta^2 \psi &- \sqrt{\text{Pr}} \beta(R) \Delta \psi = \beta(R) \Delta \psi, \\
- \sqrt{\text{Pr}} \beta(R) &+ \Delta \theta = \beta(R) \theta, \\
\psi &= \frac{\partial \psi}{\partial n} = \theta = 0 \text{ on } \partial \Omega.
\end{align*}$$

(10)

The linear equations (7) satisfy several discrete symmetries, which may be found from the known non-trivial groups of continuous Lie symmetries of the field equations (1); see Hydon [16] and Marques et al. [17]. However, for the problem we consider, it can be easily verified that the linear equations have reflection symmetries about the horizontal and vertical mid-planes of the domain. Thus, we can classify the solutions of the linear problem into four classes with different parities as defined in Table I where, for example, $\psi(o, e)$ means that $\psi$ is odd in the $x$-direction and even in the $z$-direction.

We will employ a Legendre–Galerkin method (cf. Shen [18]; Shen et al. [19]) to solve the linear eigenvalue problem (10). For this, first, we transform the domain with the change of variable

$$(x, z) \in (0, L) \times (0, 1) \mapsto (x', z') = \left(\frac{2x}{L} - 1, 2z - 1\right) \in (-1, 1)^2.$$  

For notational simplicity, we will omit the primes and write $(x, z) \in (-1, 1)^2$. The approximate solutions $(\psi^N, \theta^N)$ of (10) will be sought in the finite dimensional space, which is the span of

$$\{e_j(x)e_k(z), f_j(x)f_m(z) \mid j, l = 0, \ldots, N_x - 1, k, m = 0, \ldots, N_z - 1\},$$

where $e_j$ and $f_j$ are generalized Jacobi polynomials (cf. Guo et al. [20]; Shen et al. [19]), which satisfy the boundary conditions

$$e_j(\pm 1) = D e_j(\pm 1) = f_j(\pm 1) = 0.$$  

Here, $D$ denotes the derivative. The polynomials $e_j$ and $f_j$ are defined as in chapter 6 of Shen et al. [19],

$$f_j(z) = L_j(z) - L_{j+2}(z),$$

(11)

$$e_j(z) = \frac{L_j(z) - \frac{2j+1}{2j+3} L_{j+2}(z) + \frac{2j+1}{2j+5} L_{j+4}(z)}{(2j+3)(4j+10)^{1/2}},$$

(12)

where $L_i$ is the $i$th Legendre polynomial. The coefficient of $e_j$ guarantees that $(D^2 e_j, D^2 e_j) = \delta_{ij}$.

We write the approximate solutions of equation (10) with coefficients to be determined by

$$\psi^N = \sum_{j=0}^{N_x-1} \sum_{k=0}^{N_z-1} \tilde{\psi}^N_{jk} e_j(x) e_k(z), \quad \theta^N = \sum_{j=0}^{N_x-1} \sum_{k=0}^{N_z-1} \tilde{\theta}^N_{jk} f_j(x) f_k(z).$$

(13)

Here, $N = 2N_x N_z$ denotes the total degrees of freedom.

Let us define for $i, j = 0, \ldots, m - 1$,

$$(A^e_{ij})_{ij} = (D^2 e_i, D^2 e_j) = \delta_{ij}, \quad (A^f_{ij})_{ij} = (D^2 f_i, D^2 f_j) = -(D e_j, D e_i),$$

$$(A^e_{ij})_{ij} = (e_i, e_j), \quad (A^f_{ij})_{ij} = (e_i, f_j),$$

$$(A^e_{ij})_{ij} = (D^2 f_i, f_j), \quad (A^f_{ij})_{ij} = (f_i, f_j),$$

$$(A^f_{ij})_{ij} = (D f_i, e_j),$$

and for $j = 0, \ldots, N_x - 1, k = 0, \ldots, N_z - 1, \tilde{\psi}^N = \{\tilde{\psi}^N_{jk}\}, \tilde{\theta}^N = \{\tilde{\theta}^N_{jk}\}.$
By using the following property of the Legendre polynomials

\[(2i + 3)L_{i+1} = D(L_{i+2} - L_i),\]  

it is easy to see that

\[D_l e_i(z) = \frac{L_{i+3} - L_{i+1}}{\sqrt{4l + 10}}, \quad D^2_l e_i(z) = \sqrt{\frac{2l + 5}{2}} L_{i+2} e_i(z), \quad D_l = -(2i + 3)L_{i+1}.\]  

By (15), it is easy to determine the elements of the matrices \(A_k\) by the orthogonality of the Legendre polynomials. In particular, the matrices \(A_1^n, \ldots, A_5^n\) are banded, and except for \(A_1^n\) and \(A_5^n\), they are symmetric.

Putting (13) into (10), multiplying the resulting equations by \(e_m(x)e_n(z)\) and \(f_n(x)f_{n'}(z)\), respectively, and integrating over \(-1 \leq x \leq 1, -1 \leq z \leq 1\) to obtain

\[B^N(x) - \sqrt{RC^N(x)} = \beta^N(r)D^N(x).\]  

Here,

\[B^N = \begin{bmatrix} \Pr X_1 & 0 & 0 \end{bmatrix}_{N \times N}, \quad C^N = \begin{bmatrix} 0 & \sqrt{\Pr} X_2^T \\ -\sqrt{\Pr} X_2 & 0 \end{bmatrix}_{N \times N},\]

\[D^N = \begin{bmatrix} X_4 & 0 & 0 \end{bmatrix}_{N \times N}, \quad X^N = \begin{bmatrix} \vec{v}(\psi) \\ \vec{v}(\theta) \end{bmatrix}_{N \times 1},\]

where

\[X_1 = \frac{2^4}{L^4} A_{1}^N \otimes A_{1}^N + \frac{2^5}{L^5} A_{2}^N \otimes A_{2}^N + 2^4 A_{1}^N \otimes A_{3}^N,\]

\[X_2 = \frac{2^2}{L^2} A_{3}^N \otimes (A_{1}^N)^T, \quad X_3 = \frac{2^2}{L^2} A_{4}^N \otimes A_{2}^N + 2^2 A_{2}^N \otimes A_{3}^N,\]

\[X_4 = \frac{2^2}{L^2} A_{4}^N \otimes A_{1}^N + 2^2 A_{3}^N \otimes A_{3}^N, \quad X_5 = A_{5}^N \otimes A_{5}^N.\]  

In (17) and (18), we use the following notations. For an \(m \times k\) matrix \(M\), \(\text{vec}(M)\) is the \(mk \times 1\) column vector obtained by concatenating the columns \(M_i\) of \(M\), that is,

\[\text{vec} \left( \begin{bmatrix} M_1 & M_2 & \cdots & M_k \end{bmatrix} \right) = \begin{bmatrix} M_1^T \\ M_2^T \\ \vdots \\ M_k^T \end{bmatrix}^T.\]

0 stands for the zero matrix. If \(A\) is an \(m \times n\) matrix and \(B\) is \(p \times q\), then \(A \otimes B = \{a_{ij}b_{kl}\}\), the Kronecker product of \(A\) and \(B\), is an \(mp \times nq\) matrix defined as

\[A \otimes B = \begin{bmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \ldots & a_{mn}B \end{bmatrix}.\]

To obtain (16), we used the following properties of the Kronecker product.

- If \(A, B\), and \(X\) are three matrices such that \(AXB\) is defined, then

\[\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X).\]

- \((A \otimes B)^T = A^T \otimes B^T.\)

We note that the matrices \(B^N, C^N\), and \(D^N\) in (16) are sparse, \(B^N\) and \(D^N\) are symmetric, and \(C^N\) is skew-symmetric.

From our linear analysis, we find the following results.

- Our numerical analysis suggest that \(N_x = 6 + 2k \approx 6 + 2L\) and \(N_z = 8\) is enough to resolve the critical Rayleigh number and the first critical mode, which has \(k\) rolls in its stream function. We have checked that increasing \(N_x\) and \(N_z\) by two only modifies the fourth or fifth significant digit of the result.

- In Figure 1, the first critical mode is shown for the length scales \(L = 1, \ldots, 4\). Note that the first critical stream function and the temperature distribution have always even parity in the \(z\)-direction while their \(x\)-parity alternates between odd and even as the length scale increases. As observed by Mizushima [5], we also verify the existence of the Moffatt vortices on the corners of the domain, which are due to corner singularities as shown in Figure 2.

- For \(L < 21\), we observed that \(m\) in (9) is either 1 or 2. Moreover, \(m = 2\) only at the critical length scales, which are given in Table II. The results found are in agreement with those in the study of Mizushima [5] and Lee et al. [4].

- The marginal stability curves of the first few critical eigenvalues are given in Figure 3. The figure demonstrates that the parities of the first two critical modes can only be of parity class 1 or 2 as given in Table I.
Figure 1. $\psi$ (on top) and $\theta$ (on bottom) of the first critical mode for $L = 1, \ldots, 4$.

Figure 2. The left figure shows the plot of the first critical stream function for $L = 1$ (the top left plot in Figure 1). The right figure shows the corner details of the left figure.

Figure 3. The marginal stability curves of the first few eigenvalues with eigenfunctions of different parity classes.

Also it is seen in these figures that there is a repulsion of the eigenvalues. Namely, the neutral stability curves of the same parity type do not intersect each other. Such a repulsion does not occur for free-slip boundary conditions. This repulsion arises from a structural instability of the transform of matrices into a Jordan canonical form, and a detailed analysis can be found in the study of Mizushima and Nakamura [21].

4. Main theorem

Let $m$ be the number of modes, which become critical as the first Rayleigh number $R_{c}$ is crossed as given by (9). Ma and Wang [7, 8] proved that under some general boundary conditions, the problem has an attractor $\Sigma_{\epsilon}$, which bifurcates from $(0, R_{c})$ as $R$ crosses $R_{c}$. They also proved that the dimension of the bifurcated attractor is $m - 1 \leq \dim(\Sigma_{\epsilon}) \leq m$. When $m = 1$, the structure of $\Sigma_{\epsilon}$ is trivial, which is merely a disjoint union of two attracting steady states.

As stated before, in our problem, $m$ is either 1 or 2. And the latter case can only happen at a critical length scale $L_{c}$ where two eigenmodes with consecutive wave numbers become critical.

Numerically, it turns out that the critical Rayleigh numbers for modes with parity 3 or 4 are much greater than those for modes with parity 1 or 2. This can be seen from Figure 3.
Here, topological equivalency. Then, we have the following characterization of is sufficiently close to a critical length scale meaning that they are superpositions of the modes with wave numbers

Under the assumptions (19) and (22), there is an attractor

\[ c \]

where

Finally, let us define the following.

\[ \varphi_i = (-1)^i \phi_i, \quad i = 1, 2, \]

\[ \varphi_i = (-1)^i \phi_i, \quad i = 3, 4, \]

\[ \varphi_i = c_i \phi_1 + d_i \phi_2, \quad i = 5, 6, 7, 8, \]

Thus, \( \varphi_1, \varphi_2 \) are modes with wave number \( k, \) \( \varphi_3 \) and \( \varphi_4, \) ... , \( \varphi_8 \) are mixed modes meaning that they are superpositions of the modes with wave numbers \( k + 1. \)

**Theorem 4.1**

Under the assumptions (19) and (22), there is an attractor \( \Sigma_R \) bifurcating as \( R \) crosses \( R_c, \) which is homeomorphic to the circle \( S^1 \) when \( L \) is sufficiently close to a critical length scale \( L_c. \) Moreover, \( \Sigma_R \) consists of steady states and their connecting heteroclinic orbits. Let \( n(\Sigma_R) \) denote the number of steady states on \( \Sigma_R, \) \( S^1 \) denote the stable steady states, and \( \mathcal{U} \) denote the unstable steady states on \( \Sigma_R \) up to topological equivalency. Then, we have the following characterization of \( \Sigma_R, \) which is also illustrated in Figure 4.

(i) If \( D_1 < 0, D_2 < 0, D_3 < 0, n(\Sigma_R) = 8, S = \{ \varphi_i | i = 1, 2, 3, 4 \}, \mathcal{U} = \{ \varphi_i | i = 5, 6, 7, 8 \}. \)

(ii) If \( D_1 > 0, D_2 > 0, D_3 > 0, n(\Sigma_R) = 8, S = \{ \varphi_i | i = 5, 6, 7, 8 \}, \mathcal{U} = \{ \varphi_i | i = 1, 2, 3, 4 \}. \)

(iii) If \( D_1 < 0, D_2 < 0, n(\Sigma_R) = 4, S = \{ \varphi_i | i = 1, 2 \}, \mathcal{U} = \{ \varphi_i | i = 3, 4 \}. \)

(iv) If \( D_1 > 0, D_2 < 0, n(\Sigma_R) = 4, S = \{ \varphi_i | i = 3, 4 \}, \mathcal{U} = \{ \varphi_i | i = 1, 2 \}. \)
According to Theorem 4.1, the structure of the attractor depends on $D_1$, $D_2$, and $D_3$, which in turn depends on the coefficients of the reduced equations. By (21), $D_3$ has a definite sign whereas $D_1$ and $D_2$ vanish at the criticality $\beta_1 = \beta_2 = 0$. In the proof of Theorem 4.1, we analytically prove that the coefficients $a_{11}$ and $a_{24}$ are negative. Our numerical computations indicate that $a_{13}$ is also always negative. We observed that $a_{22}$ and $D_3$ can be both positive and negative. That gives three possible cases depending on the signs of $a_{22}$ and $D_3$. In Figure 5, we classify these cases in a small neighborhood of $\beta_1 - \beta_2$ plane according to our main theorem and the following observations.

- If $a_{22} > 0$, then $D_3 > 0$, $D_1 > 0$, but $D_2$ changes sign in the first quadrant.
- If $a_{22} < 0$ and $D_3 > 0$, then $D_1$ and $D_2$ change sign in the first quadrant. Moreover, the case where both $D_1 < 0$ and $D_2 < 0$ is not possible.
- If $a_{22} < 0$ and $D_3 < 0$, then again $D_1$ and $D_2$ change sign in the first quadrant. This time the case where both $D_1 > 0$ and $D_2 > 0$ is not possible.

5. Proof of the main theorem

We will give the proof in several steps.

Step 1. The reduced equations. When there are two critical modes $\phi_1, \phi_2$, the center manifold is a 2D manifold embedded in the infinite dimensional space. We denote the center manifold function by

$$
\Phi = y_1^{2} \Phi_1 + y_1 y_2 \Phi_2 + y_2^{2} \Phi_3 + o(y^2), \quad \Phi_i = (U_i, W_i, \Theta_i).
$$
To study the dynamics on the center manifold, we plug in
\[ \phi = y_1\phi_1 + y_2\phi_2 + y_1^2\Phi_1 + y_1y_2\Phi_2 + y_2^2\Phi_3 + o(2), \] (24)
into (5), take the inner product with \(\phi_1, \phi_2\), and use the orthogonality of the eigenvectors, thanks to the self-adjointness of the linear operator. The reduced equations read
\[ \frac{dy_i}{dt} = \beta_i(R)y_i + \frac{1}{(\phi_i, \phi_i)}(G(\phi_i), \phi_i), \quad i = 1, 2 \] (25)
We normalize the first two eigenfunctions so that \((\phi_1, \phi_1) = (\phi_2, \phi_2) = 1\).
Now if we expand the nonlinear terms in (25), we obtain
\begin{align*}
\frac{dy_1}{dt} &= \beta_1y_1 + (a_{11}y_1^2 + a_{12}y_1y_2 + a_{13}y_1^2y_2 + a_{14}y_1^2) + o(3), \\
\frac{dy_2}{dt} &= \beta_2y_2 + (a_{21}y_1^2 + a_{22}y_1^2y_2 + a_{23}y_1y_2^2 + a_{24}y_1^2) + o(3),
\end{align*}
(26)
where
\begin{align*}
a_{k1} &= G_s(\phi_1, \Phi_1, \Phi_k), \quad a_{k2} = G_s(\phi_1, \Phi_2, \Phi_k) + G_s(\phi_2, \Phi_1, \Phi_k), \\
a_{k3} &= G_s(\phi_2, \Phi_3, \Phi_k), \quad a_{k4} = G_s(\phi_1, \Phi_3, \Phi_k) + G_s(\phi_2, \Phi_2, \Phi_k).
\end{align*}
(27)

**Step 2. Parities of the center manifold functions.** To compute the center manifold approximation, we will use the following formula, which was introduced by Ma and Wang [15].
\[ -\mathcal{L}\Phi_1 = P_2G(\phi_1, \phi_1), \quad -\mathcal{L}\Phi_2 = P_2[G(\phi_1, \phi_2) + G(\phi_2, \phi_1)], \quad -\mathcal{L}\Phi_3 = P_2G(\phi_2, \phi_2). \] (28)
Here,
\[ P_2 : H \to E_2, \quad \mathcal{L}_E = \mathcal{L}_E|_{E_2} : E_2 \to \check{E}_2, \quad E_1 = \text{span}\{\phi_1, \phi_2\}, \quad E_2 = E_1^\perp. \]
Let \(\mathcal{X} = \{f \in C(\Omega) \mid f(-x, z) = \pm f(x, z)\text{ and }f(x, -z) = \pm f(x, z)\}\), and let \(s : \mathcal{X} \to \{\pm 1\}^2\) denote the parity function:
\[ s(f) = (s_x(f), s_z(f)), \]
where
\[ s_x(f) = \pm 1 \text{ if } f(-x, z) = \pm f(x, z), \quad s_x(f) = \pm 1 \text{ if } f(x, -z) = \pm f(x, z). \]
Let us define for \(\phi_i = (u_i, w_i, \theta_i), i = 1, 2\), the following.
\[ G(\phi_i, \phi_j) = \begin{bmatrix} g_1(\phi_i, \phi_j) \\ g_2(\phi_i, \phi_j) \\ g_3(\phi_i, \phi_j) \end{bmatrix} = \begin{bmatrix} -u_i \frac{\partial w_i}{\partial \theta_i} - w_i \frac{\partial u_i}{\partial \theta_i} \\ -u_i \frac{\partial w_i}{\partial \theta_i} - w_i \frac{\partial u_i}{\partial \theta_i} \\ -u_i \frac{\partial w_i}{\partial \theta_i} - w_i \frac{\partial u_i}{\partial \theta_i} \end{bmatrix}. \] (29)
The following lemma can be proven using the basic properties of parities.

**Lemma 5.1**
If \(\phi_i = (u_i, v_i, \theta_i) \in \mathcal{X}^3 \cap H_1, i = 1, 2\), then for \(i, j, k = 1, 2\),
1. \(-s(g_1(\phi_i, \phi_k)) = s(g_2(\phi_i, \phi_k)) = s(g_3(\phi_i, \phi_k)) = (s_x(w_i w_j), -s_z(w_i w_j)). \)
2. \(s(g_k(\phi_i, \phi_k)) = s(g_k(\phi_i, \phi_k)). \)

Hereafter, without loss of generality, we will assume that \(\phi_1\) is of parity class 1 and \(\phi_2\) is of parity class 2,
\[ \phi_1 \text{ is of parity class 1 and } \phi_2 \text{ is of parity class 2}, \] (30)
which are as given in Table I.
Using Lemma 5.1, we can prove

**Lemma 5.2**
Under the assumption (30),
\[ s(g_2(\phi_1, \phi_1)) = s(g_2(\phi_2, \phi_2)) = (1, -1), \quad s(g_2(\phi_1, \phi_2)) = (-1, -1). \]

**Lemma 5.3**
Under the assumption (30), \(P_2G(\phi_i, \phi_i) = G(\phi_i, \phi_i)\) for \(i, j = 1, 2\).
### Proof

Note that $P_2 G(\phi, \phi_k) = G(\phi, \phi_k)$ if $(G(\phi, \phi_j), \phi_k) = 0$ for $i, j, k = 1, 2$. Now,

$$
(G(\phi, \phi_j), \phi_k) = \int_{\Omega} (g_1(\phi, \phi_j) u_k + g_2(\phi, \phi_j) w_k + g_3(\phi, \phi_j) \theta_k) \, dx dz.
$$

By Lemma 5.1 and Lemma 5.2, $g_1(\phi, \phi_j)$ is even in the $z$-direction while $g_2(\phi, \phi_j)$ and $g_3(\phi, \phi_j)$ are odd in the $z$-direction. Because $u_k$ is odd and $w_k$ and $\theta_k$ are even in the $z$-direction, the integral in (31) must vanish over $\Omega$.

Thus, by Lemma 5.3 and equation (28), $\Phi_i = (U_i, W_i, \Theta_i)$, ($i = 1, 2, 3$) are solutions of

$$
\begin{align*}
-L_0 \Phi_1 &= G(\phi_1, \phi_1), \\
-L_0 \Phi_2 &= G(\phi_2, \phi_2) + G(\phi_2, \phi_1), \\
-L_0 \Phi_3 &= G(\phi_2, \phi_2).
\end{align*}
$$

Using the stream function $\Psi_x = U$, $\Psi_z = -W$, one can eliminate the pressure from these equations to obtain

$$
\begin{align*}
\text{Pr} \Delta^2 \Psi - \sqrt{\text{Pr}} \sqrt{\text{Pr}} \frac{\partial \Theta}{\partial x} &= h_1 := -\frac{\partial g_1}{\partial z} + \frac{\partial g_2}{\partial x}, \\
-\sqrt{\text{Pr}} \sqrt{\text{Pr}} \frac{\partial \Psi}{\partial x} + \Delta \Theta &= h_2 := -g_3, \\
\Psi &= \frac{\partial \Psi}{\partial n} = \Theta = 0 \text{ on } \partial \Omega.
\end{align*}
$$

**Lemma 5.4**

Under the assumption (30), the center manifold functions have the parity as given in Table III.

**Proof**

We can eliminate $\Theta$ from the first equation of (33) to obtain

$$
\begin{align*}
\text{Pr} \Delta^2 \Psi - \text{Pr} \frac{\partial^2 \Psi}{\partial x^2} &= \Delta h_1 + \sqrt{\text{Pr}} \sqrt{\text{Pr}} \frac{\partial h_2}{\partial x}, \\
\Delta \Theta &= h_2 + \sqrt{\text{Pr}} \sqrt{\text{Pr}} \frac{\partial \Psi}{\partial x}.
\end{align*}
$$

Now using Lemma 5.1 and Lemma 5.2, we see that $s(\Psi) = (-s_x(g_2), s_z(g_2))$ and $s(\Theta) = s(g_2)$.

Using Table III, we find that the integrands in $a_{12}, a_{14}, a_{21}, a_{23}$ are all odd functions of $z$, and hence, we have the following.

**Lemma 5.5**

Under the assumption (30), in (27), we have

$$
a_{12} = a_{14} = a_{21} = a_{23} = 0.
$$

As a result of Lemma 5.5, we obtain the reduced equations (20).

**Step 3. The attractor bifurcation.** Now, we will prove that the bifurcated attractor is homeomorphic to $S^1$. For this, we will need the following result.

**Theorem 5.6 (Ma and Wang [15])**

Let $v$ be a 2D $C^1$ ($r \geq 1$) vector field given by

$$
v_\lambda(x) = \beta(\lambda) x - h(x, \lambda),
$$

for $x \in \mathbb{R}$. Here, $\beta(\lambda)$ is a continuous function of $\lambda$ satisfying $\beta(\lambda) \equiv 0$ for $\lambda \geq \lambda_0$ and

$$
h(x, \lambda) = h_k(x, \lambda) + o(|x|^k), \quad C_1 |x|^{k+1} \leq (h_k(x, \lambda), x),
$$

for some odd integer $k \geq 3$ where $h_k(\cdot, \lambda)$ is a $k$-multilinear field, and $C_1 > 0$ is some constant. Then, the system

$$
dx/dt = v_\lambda(x), \quad x \in \mathbb{R},
$$

bifurcates from $(x, \lambda) = (0, \lambda_0)$ to an attractor $\Sigma_\lambda$, which is homeomorphic to $S^1$, for $\lambda_0 < \lambda < \lambda_0 + \epsilon$, for some $\epsilon > 0$. Moreover, either (i) $\Sigma_\lambda$ is a periodic orbit, or (ii) $\Sigma_\lambda$ consists of an infinite number of singular points, or (iii) $\Sigma_\lambda$ contains at most $2(k+1)$ singular points, consisting of $2N$ saddle points, $2N$ stable node points, and $n(2(k+1) - 4N)$ singular points with index zero.
Now let
\[ h(y_1, y_2) = \begin{bmatrix} y_1 (a_{11} y_1^2 + c_{12} y_2^2) + y_2 (a_{22} y_1^2 + c_{22} y_2^2) \end{bmatrix}^T. \]

**Lemma 5.7**
Assume that \( \Phi_j \neq 0 \) for \( i = 1, 2, 3 \). Then for any \( y = (y_1, y_2) \),
\[
(h(y), y) = a_{11} y_1^4 + (a_{12} + a_{22}) y_1^2 y_2^2 + a_{22} y_2^4 \leq C |y|^4,
\]
where \( C < 0 \).

**Proof**
\[
a_{11} = G_1(\Phi_1, \Phi_1, \Phi_1) = G(\Phi_1, \Phi_1, \Phi_1) + G(\Phi_1, \Phi_1, \Phi_1)
= G(\Phi_1, \Phi_1, \Phi_1) = -G(\Phi_1, \Phi_1, \Phi_1) = (\mathcal{L}_R \Phi_1, \Phi_1).
\]
Here, we used (32) and the following properties of the Navier–Stokes nonlinearity
\[
(i) \ G(\Phi, \Phi^*, \Phi^*) = G(\Phi, \Phi^*, \Phi^*), \quad (ii) \ G(\Phi, \Phi^*, \Phi^*) = 0,
\]
and \(-\mathcal{L}_R \Phi_1 = G(\Phi_1, \Phi_1)\), which is due to (32).

If we write
\[
\Phi_j = \sum_{k=3}^{\infty} c_{jk} \Phi_k, \quad j = 1, 2, 3,
\]
then for \( j = 1, 2, 3 \),
\[
(\mathcal{L}_R \Phi_j, \Phi_j) = \sum_{k=3}^{\infty} c_{jk}^2 \beta_k ||\Phi_k||^2 < 0.
\]
Because \( \beta_k < 0 \) for \( k \geq 3 \) and by assumption, there exists \( k \geq 3 \) such that \( c_{jk} \neq 0 \). In particular, \( a_{11} < 0 \). As in (37), we can show that
\[
a_{24} = G_1(\Phi_3, \Phi_3, \Phi_2) = (\mathcal{L}_R \Phi_3, \Phi_3) < 0.
\]

Now if \( a_{13} + a_{22} < 0 \), then it is easy to prove (36). Assume otherwise. Using (38) and (32), we can write
\[
a_{13} = G_1(\Phi_1, \Phi_2, \Phi_1) + G_1(\Phi_1, \Phi_2, \Phi_1)
= G(\Phi_1, \Phi_3, \Phi_1) + G(\Phi_3, \Phi_1, \Phi_1) + G_1(\Phi_2, \Phi_2, \Phi_1)
= -G(\Phi_1, \Phi_3) + G_1(\Phi_2, \Phi_2, \Phi_1)
= (\mathcal{L}_R \Phi_1, \Phi_3) + G_1(\Phi_2, \Phi_2, \Phi_1).
\]
A similar computation shows
\[
a_{22} = (\mathcal{L}_R \Phi_1, \Phi_3) + G_1(\Phi_3, \Phi_2, \Phi_2).
\]
Let us define
\[
\alpha = G_1(\Phi_1, \Phi_2, \Phi_1).
\]
By (38) and (32),
\[
\alpha = -G(\Phi_1, \Phi_2) + G(\Phi_2, \Phi_1) = (\mathcal{L}_R \Phi_2, \Phi_2).
\]
Note that \( \alpha < 0 \) by (39). By using Cauchy–Schwarz inequality and the orthogonality of the eigenfunctions,
\[
(\mathcal{L}_R \Phi_1, \Phi_3) = \sum_{k=3}^{\infty} \beta_k c_{12} c_{3k} ||\Phi_k||^2 \leq \left( \sum_{k=3}^{\infty} -\beta_k c_{12}^2 ||\Phi_k||^2 \right)^{1/2} \left( \sum_{k=3}^{\infty} -\beta_k c_{3k}^2 ||\Phi_k||^2 \right)^{1/2}
= \sqrt{a_{11} a_{24}}.
\]
Because \( \mathcal{L}_R \Phi_1, \Phi_3 = (\mathcal{L}_R \Phi_1, \Phi_3) \), we have by (40)–(44),
\[
a_{13} + a_{22} < 2 \sqrt{a_{11} a_{24}} + \alpha,
\]
where \( \alpha < 0 \) is given by (42). Thus, there exists \( 0 < \epsilon_1 < -a_{11}, 0 < \epsilon_2 < -a_{24} \) such that
\[
a_{13} + a_{22} < 2 \sqrt{a_{11} a_{24}} + \alpha < 2 \sqrt{(a_{11} + \epsilon_1)(a_{24} + \epsilon_2)}.
\]
Because \( 2ab < a^2 + b^2 \), we have
\[
2 \sqrt{(a_{11} + \epsilon_1)(a_{24} + \epsilon_2)} y_1^2 y_2^2 \leq -(a_{11} + \epsilon_1) y_1^4 - (a_{24} + \epsilon_2) y_2^4.
\]
Now, let \( C = \max\{-\epsilon_1, -\epsilon_2\} \). Then, \( C < 0 \), and we have
\[
(h(y), y) \leq a_{11}y_1^4 + (a_{13} + a_{22})y_1^2y_2^2 + a_{24}y_2^4 \leq C(x^2 + y^2)^2.
\]

That finishes the proof. \( \square \)

Thus, by Theorem 5.6 and Lemma 5.7, \( \Sigma_S \) is homeomorphic to \( S^1 \). Now we will describe the details of its structure by determining the bifurcated steady states and their stabilities.

**Step 4. The steady states and their stabilities.** The possible equilibrium solutions of the truncated equations of (20) are as follows.

\[
R_1 = \left( \sqrt{\frac{b_1}{a_{11}}}, 0 \right), \quad R_2 = \left( 0, \sqrt{\frac{b_2}{a_{24}}} \right), \quad M = \left( \frac{D_2}{D_1}, \frac{D_1}{D_2} \right).
\]

where \( D_1, D_2, \) and \( D_3 \) are given by (21).

Because of the invariance of the equations (20) with respect to \((x, y) \rightarrow (-x, y)\) and \((x, y) \rightarrow (x, -y)\), we only consider the positive solutions when writing (45).

The eigenvalues \( \lambda_1, \lambda_2 \) of the truncated vector field at the steady states \( R_1, R_2 \) are
\[
\lambda_1 = -2\beta_1, \quad \lambda_2 = -D_1/a_{11}, \quad \lambda_1^* = -2\beta_2, \quad \lambda_2^* = -D_2/a_{24}.
\]

Note that \( R_i \) is always bifurcated for \( \beta_i > 0, i = 1, 2 \). Moreover, \( R_i \) is a stable steady state for \( \beta_i > 0 \) if \( D_i < 0 \) for \( i = 1, 2 \). The trace \( \text{Tr} \) and the determinant \( \text{Det} \) of the Jacobian matrix of the truncated vector field at the mixed states \( M \) are
\[
\text{Tr} = \frac{2}{D_3}(a_{24}D_1 + a_{11}D_2), \quad \text{Det} = \frac{4}{D_3}D_1D_2.
\]

Notice that the steady states \( M \) are bifurcated only when \( D_1, D_2, D_3 \) have the same sign. Because \( a_{11} \) and \( a_{24} \) are both negative as shown in Lemma 5.7, according to trace-determinant plane analysis, they are saddles if \( D_1 < 0, D_2 < 0, D_3 < 0 \) and are stable if \( D_1 > 0, D_2 > 0, D_3 > 0 \).

Finally, only the four cases stated in our main theorem can occur. To see this, note that according to Theorem 5.6 and (45)–(46), the case \( D_1 < 0, D_2 < 0, D_3 > 0 \) is not possible because that would lead to only four steady states on the attractor, which are all stable. Similarly, the case \( D_1 > 0, D_2 > 0, D_3 < 0 \) is not possible either, which would lead to four steady states, which are all unstable.

### 6. Numerical approximation of the coefficients of the reduced equations

To compute the coefficients of the reduced equations (20), we fix \( L, Pr, \) and \( R \) to compute all the eigenvalues \( \beta_i^N \) and the corresponding eigenvectors of (16).

**Numerical computation of the center manifold functions.** Now we will numerically approximate \( \Phi_1, \Phi_2, \) and \( \Phi_3 \), which are the solutions of the equations (28). We will illustrate the method to approximate \( \Phi_1 \), because \( \Phi_2, \Phi_3 \) can be approximated similarly. To determine \( \Phi_1 \), we have to find its stream function \( \Psi \) and its temperature function \( \Theta \), which are determined by the equations (33).

Because we do not have \( h_1 \) and \( h_2 \) in (33) exactly, we approximate them by \( h_1^N, h_2^N \) as that shown later.

\[
\begin{align*}
h_1^N &= \frac{\partial g_1^N}{\partial z} + \frac{\partial g_2^N}{\partial x}, \\
g_1^N &= -\psi_{1,2}^N \psi_{1,2}^N + \psi_{1,1}^N \psi_{1,22}^N, \\
\end{align*}
\]

(47)

Here, \( (\psi_1^N, \partial_1^N) \) is the first critical eigenfunction of the discrete problem (16).

\[
\{ \psi_1^N, \partial_1^N \} = \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_y-1} \left\{ \tilde{\psi}_{1,mn}^N e_m(x) e_n(z), \ \tilde{\partial}_{1,mn}^N f_m(x) f_n(z) \right\}.
\]

(48)

**The Legendre–Galerkin approximation of the problem (33).** As in the linear eigenvalue problem, we discretize the equations (33) using the generalized Jacobi polynomials (11)–(12).

\[
\{ \psi^N, \Theta^N \} = \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_y-1} \left\{ \tilde{\psi}_{m,n}^N e_m(x) e_n(z), \ \tilde{\Theta}_{m,n}^N f_m(x) f_n(z) \right\}.
\]

(49)

We plug in \( \Psi^N, \Theta^N, h_1^N, h_2^N \) for \( \psi, \Theta, h_1, h_2 \) in (33) and multiply the resulting equations by Jacobi polynomials \( e_i(x) e_k(z), f_i(x) f_k(z) \) and integrate over \(-1 \leq x \leq 1, -1 \leq z \leq 1\) to reduce (33) to the following finite dimensional linear equation

\[
\left( B^N - \sqrt{R} C^N \right) \tilde{x} = \tilde{b}.
\]

(50)
Here, $B^N$ and $C^N$ are given by (17) and

$$\tilde{x} = \left[ \text{vec} \left( \hat{\psi}^N \right) \right]_{N \times 1}^T, \quad \tilde{b} = \left[ \text{vec}(B_1) \text{ vec}(B_2) \right]^T. \quad (51)$$

For $0 \leq j \leq N_x - 1, 0 \leq k \leq N_z - 1,$

$$\begin{align*}
(B_1)_{jk} &= \int_{-1}^{1} \int_{-1}^{1} h^N_i(x, z) \xi_j(x) \xi_k(z) dxdz, \\
(B_2)_{jk} &= \int_{-1}^{1} \int_{-1}^{1} h^N_j(x, z) \xi_k(x) \xi_k(z) dxdz.
\end{align*} \quad (52)$$

Now, $\xi_j$ is a polynomial of degree $j + 4$, and by (47) and (48), $h^N_i$ is a polynomial of degree at most $(2N_x + 6, 2N_z + 6)$. Thus, the aforementioned integrands are of degree at most $(3N_x + 9, 3N_z + 9)$. Because the Legendre–Gauss–Lobatto quadrature with $N + 1$ quadrature points is exact for polynomials of degree less or equal than $2N - 1$, the integrals in (52) can be replaced by the following discrete inner products.

$$\begin{align*}
(B_1)_{jk} &= \frac{1}{2} \sum_{m=0}^{\frac{1}{2}N_x + 5} \sum_{n=0}^{\frac{1}{2}N_z + 5} h^N_i(x_m, z_n) \xi_j(x_m) \xi_k(z_n) \omega_m \omega_n, \\
(B_2)_{jk} &= \frac{1}{2} \sum_{m=0}^{\frac{1}{2}N_x + 5} \sum_{n=0}^{\frac{1}{2}N_z + 5} h^N_j(x_m, z_n) \xi_k(x_m) \xi_k(z_n) \omega_m \omega_n. \quad (53)
\end{align*}$$

Here, $\{x_m, w^N_m\}_{j=0}^{\frac{1}{2}N_x + 5}$ and $\{z_n, w^N_n\}_{j=0}^{\frac{1}{2}N_z + 5}$ are the Legendre–Gauss–Lobatto points and weights in the $x$-direction and the $z$-direction.

**Solution of (50).** The solution $\tilde{x}$ of (50) can be obtained by inverting the matrix $\left( B^N - \sqrt{R} C^N \right)$. But this matrix has a large condition number. Thus, we show a method to obtain the solution inverting the matrix $D^N$ given by (17), which has a much smaller condition number. For example, for $N_x = 10, N_z = 8$, the condition number of $\left( B^N - \sqrt{R} C^N \right)$ is $O(10^{16})$ while the condition number of $D^N$ is $O(10^8)$.

Because $\Phi_i \in E_2 = \text{span}\{\phi_1, \phi_2\}^\perp$, we look for a solution of (50) in the form

$$\tilde{x} = \sum_{i=3}^{N} x_i \tilde{x}_i, \quad (54)$$

where $\tilde{x}_i$ are the eigenvectors of

$$B^N \tilde{x}_i - \sqrt{R} C^N \tilde{x}_i = \beta_i(R) D^N \tilde{x}_i. \quad (55)$$

If we multiply (50) by $(D^N)^{-1}$ and use (55), the left-hand side of (50) becomes

$$\sum_{i=3}^{N} x_i \beta_i(R) \tilde{x}_i = (D^N)^{-1} \left( B^N - \sqrt{R} C^N \right) \tilde{x} = (D^N)^{-1} \tilde{b} := \tilde{f}. \quad (56)$$

We determine $\tilde{f}$ from $D^N \tilde{f} = \tilde{b}$ using Gaussian elimination. Once again using Gaussian elimination, we can find the coefficients $f_i$ in the expansion

$$\tilde{f} = \sum_{i=1}^{N} f_i \tilde{x}_i. \quad (57)$$

In (57), we see that $f_1 = f_2 = 0$ is necessary for the existence of a solution of (50). From (56) and (57), one finds $x_i = f_i/\beta_i(R), \quad i = 3, 4, \ldots, N$. Thus, the Jacobi expansion coefficients in (49) of the center manifold are given by

$$\begin{align*}
\text{vec} \left( \hat{\psi}^N \right) &= \sum_{i=3}^{N} \frac{f_i}{\beta_i(R)} \text{ vec } (\hat{\psi}^N), \quad \text{vec } (\hat{\phi}^N) = \sum_{i=3}^{N} \frac{f_i}{\beta_i(R)} \text{ vec } (\hat{\phi}^N).
\end{align*}$$

**Numerical computation of $a_{ij}$ in (27).** We approximate $a_{11}$ by

$$a_{11}^N = G_i \left( \phi_1^N, \Phi_1^N, \phi_1^N \right). \quad (58)$$

The integrands in $G_i(\phi_1^N, \Phi_1^N, \phi_1^N)$ are polynomials of degree at most $(3N_x + 9, 3N_z + 9)$. Thus, to replace the integrals in (58), one needs again $(\frac{1}{2}N_x + 5, \frac{1}{2}N_z + 5)$ quadrature points and nodes in the numerical inner product. The other coefficients $a_{ij}$ in (27) are approximated similarly.
We computed coefficients of the reduced equations for various Pr values ranging from 0.1 to 10^3 at the first three critical length scales and at the critical Rayleigh numbers, which are given in Table II.

We observed that increasing Pr values around (but not necessarily very close to) the criticality Pr = 0.06. Thus, the transition is as described in Figure 5(b) for Pr < 0.05 and as described in Figure 5(c) for Pr > 0.15. Thus, the mixed modes can be stable when Pr < 0.14, but only the pure modes are stable points of the attractor when Pr < 0.05; however, the mixed modes are stable steady states when Pr > 0.06. In particular, the mixed modes can be stable when Pr < 0.05, but only the pure modes are stable steady states when Pr > 0.06.

Remark 6.1

We observed that increasing N_x and N_z above N_x = 10 + 2k and N_z = 8 only changes a_{ij}^k in the seventh digit when the first critical mode has k rolls and the second critical mode has k + 1 rolls in their stream functions.

7. Numerical results and discussion

We computed coefficients of the reduced equations for various Pr values ranging from 0.1 to 10^3 at the first three critical length scales and at the critical Rayleigh numbers, which are given in Table II.

As proven in Theorem 4.1, the coefficients a_{11} and a_{22} are always negative. In our numerical calculations, we encountered that a_{13} is also always negative. But the signs of a_{22} and the sign of D_3 depend on L and Pr and are given in Table IV.

For the first critical length scale L_c = 1.5702, we found that a_{22} and D_3 change sign from positive to negative between 0.04 < Pr < 0.05 and 0.14 < Pr < 0.15, respectively. Thus, the transition is as described in Figure 5(a) for Pr < 0.04, as in Figure 5(b) for 0.05 < Pr < 0.14, and as in Figure 5(c) for Pr > 0.15. Thus, the mixed modes can be stable when Pr < 0.14, but only the pure modes are stable points of the attractor when Pr > 0.15.

For the second critical length scale L_c = 2.6611, we always observed that a_{22} < 0. However, D_3 changes sign between 0.05 < Pr < 0.06. Thus, the transition is as described in Figure 5(b) for Pr < 0.05 and as described in Figure 5(c) for Pr > 0.06. In particular, the mixed modes can be stable when Pr < 0.05, but only the pure modes are stable steady states when Pr > 0.06.

For higher critical length scales (third and beyond), we found that a_{22} < 0 and D_3 < 0 for the Prandtl numbers we considered. Thus, the transition is as described in Figure 5(c). For this length scale, either the critical Prandtl number that was observed for the first two critical length scales is now very close to zero, or it does not exist at all.

The previous analysis depends on the coefficients a_{ij} of the reduced equations and predicts the transitions when both eigenvalues \( \beta_1, \beta_2 \) are close to zero. Now, we present an analysis depending on the direct computation of the numbers D_1, D_2 (both of which vanish when \( \beta_1 = \beta_2 = 0 \)), and D_3. We computed D_1, D_2, and D_3 for L and R values around (but not necessarily very close to) the criticality (L, R) = (L_c, R_c) for the first three critical length scales and for Prandtl numbers Pr = 0.1, 0.71, 7, 130. The results are shown in Figure 6.
Although we might have omitted the smallness assumptions of $|L - L_c|$ and $|R - R_c|$ where our main theorem is valid, these figures help us predict the transitions in the $L$–$R$ plane. The results we obtain are as follows.

For $Pr = 0.71$, $Pr = 7$, $Pr = 130$, transitions are qualitatively the same in the $L$–$R$ plane. For $L > L_c$, the basic motionless state loses its stability to the eigenmode with wave number $k + 1$ as the Rayleigh number crosses the first critical Rayleigh number and further increase of the Rayleigh number does not alter the stability of this steady state. This is in contrast to the situation $L < L_c$ where there is a transition of stabilities as the Rayleigh number is increased. Namely, as the Rayleigh number crosses the first critical Rayleigh number, the eigenmode with wave number $k$ becomes stable. As the Rayleigh number is further increased, both eigenmodes coexist as stable steady states and the initial conditions determine which one of these steady states will be realized. Finally, as the Rayleigh number is further increased, the eigenmode with wave number $k + 1$ becomes stable.

The transition at $Pr = 0.1$ is essentially different than for those at $Pr = 0.71$, 7, 130. In particular, for the first critical length scale $L_c = 1.5702$, for $L < L_c$, subsequently, mode with wave number $k$, mixed modes, and finally mode with wave number $k + 1$ will be realized as the Rayleigh number is increased while for $L > L_c$, $k + 1$ mode is the only stable steady state.

Acknowledgements

The work of Shen is partially supported by NSF-DMS-1217066 and DMS-1419053, and the work of Wang is supported in part by NSF-DMS-1211218.

References