ADVANCE TOPICS

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ABSTRACT. I made this notes to preparte my "Advance Topics Exam" (a.k.a. oral exam) at Purdue University. I principally follow [Tay81] (and [Tay11]) for the theory of ΨDOs, and [SU] for the applications to X-ray transform. I use the other references to look proofs from other point of view or to expand them, and to obtain examples that always help me to understand the theory. These notes do not replace the works [Tay81] nor [SU], for example, the proofs on this document are far from being complete. The interested reader should study those references to fully understand the topic treated here.

CONTENTS

INTRODUCTION

1. ΨDOs

1.1. **ΨDOs calculus.** ΨDOs are natural generalization of differential operators. Let us give some examples to motivate its study.

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Example 1.1.

(1) Using the Fourier Inversion Formula and the properties of the Fourier Transform, we have that if $p(x,D) = \sum_{|\alpha|\le k} a_\alpha(x) D^\alpha$, then

$$p(x,D)u(x) = \int_{\mathbb{R}^n} p(x,\xi)\hat{u}(\xi)e^{ix\cdot\xi}d\xi,$$

where $p(x,\xi) = \sum_{|\alpha|\le k} a_\alpha(x)\xi^\alpha$. So, one could ask what happens is we take $p(x,\xi)$ not being a polynomial in ξ.

(2) Consider the following PDE

$$-\Delta u = f.$$

By applying the Fourier transform we obtain

$$\hat{u} = \frac{1}{|\xi|^2}\hat{f}.$$

So,

$$u = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{1}{|\xi|^2}\hat{f}(\xi)d\xi.$$

Note that we have a singularity at zero. One form to solve this, is to take a smooth function with compact support φ that is 1 in a neighborhood of zero. Then, we can write

$$\frac{1}{|\xi|^2} = \frac{1-\varphi(\xi)}{|\xi|^2} + \frac{\varphi(\xi)}{|\xi|^2},$$

So that

$$u = \mathcal{F}^{-1}\left(\frac{1-\varphi(\xi)}{|\xi|^2}\hat{f}\right) + \mathcal{F}^{-1}\left(\frac{\varphi(\xi)}{|\xi|^2}\hat{f}\right).$$

This is the base for the following definition.

Definition 1.2. Let $\Omega \subset \mathbb{R}^n$ be an open set. We define *symbol class*

$$\begin{aligned} S^m(\Omega) = \{p \in C^\infty(\Omega\times\mathbb{R}^n) : \forall K \subset \Omega \text{ compact}, \forall \alpha,\beta, \exists C_{K,\alpha,\beta} > 0 \text{ s.t.} \\ |D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{K,\alpha,\beta}(1+|\xi|)^{m-|\alpha|}\,\forall (x,\xi)\in K\times\mathbb{R}^n\}. \end{aligned}$$

We also set $S^{-\infty} := \cap_m S^m$.

Example 1.3. Let φ as in Example 1.1 (2). Then, $\frac{1-\varphi(\xi)}{|\xi|^2} \in S^{-2}$.

Theorem 1.4. *If $p \in S^m$, then $P(x,D)$ defined by*

$$P(x,D)u(x) = \int_{\mathbb{R}^n} p(x,\xi)\hat{u}(\xi)e^{ix\cdot\xi}d\xi,$$

is a continuous operator $P(x,D)\colon C_0^\infty(\Omega) \to C^\infty(\Omega)$. Furthermore, this map can be extended to a continuous map $p(x,D)\colon \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$. We write $P = \mathrm{Op}(p)$, and $P \in \Psi^m$. Furthermore, we write $\Psi^{-\infty} = \cap\Psi^m$.

Proof.

- First part: $u \in C_0^\infty(\Omega) \implies \hat{u}(\xi) = O(|\xi|^{-N})$ for any N. So

$$|P(x,D)u(x)| \le C + C_N\int_{\{|\xi|\ge 1\}}(1+|\xi|)^m|\xi|^{-N}d\xi < \infty.$$

So the integral is absolutely convergent. Same works for derivatives.

- Second part: $v \in C_0^\infty$ multiplying by η^α and integration by parts gives

$$|p_v(\xi)| := \left| \int_{\mathbb{R}^n} v(x)p(x,\xi)e^{ix\cdot\eta}dx \right| \leq C_N(1+|\xi|)^m(1+|\eta|)^{-N}.$$

Since p_v rapidly decreasing, $\langle p(x,D)u, v\rangle = \int_{\mathbb{R}^n} p_v(\xi)\hat{u}(\xi)d\xi$ is well defined.

□

Example 1.5. The operator $f \mapsto \mathcal{F}^{-1}\left(\frac{1-\varphi(\xi)}{|\xi|^2}\hat{f}\right)$ belongs to Ψ^{-2}. On the other hand, $f \mapsto \mathcal{F}^{-1}\left(\frac{\varphi(\xi)}{|\xi|^2}\hat{f}\right)$ is not a ΨDO because of the singularity on the origin.

We also have the following class of symbols

Definition 1.6. We say that $p \in S^m(\Omega)$ is a *classical symbol* if there are smooth p_{m-j}, positively homogeneous in ξ of order $m-j$ for $|\xi| \geq 1$, i.e.,

$$p_{m-j}(x,\lambda\xi) = \lambda^{m-j}p_{m-j}(x,\xi), \quad |\xi| \geq 1, \lambda > 1,$$

such that for all $N \geq 0$

$$p(x,\xi) - \sum_{j=0}^{N} p_{m-j}(x,\xi) \in S^{m-N-1}(\Omega).$$

This asymptotic condition is written as $p(x,\xi) \sim \sum_{j=0}^\infty p_{m-j}(x,\xi)$. The set of classical symbols is denoted by S^m_{cl}.

Example 1.7. An operator in $S^{1/2+\varepsilon}(\mathbb{R}^n) \setminus S^{1/2+\varepsilon}_{\mathrm{cl}}(\mathbb{R}^n)$ (with $\varepsilon > 0$) is given by the operator with symbol $\chi(\xi)|\xi|^{1/2}\ln(|\xi|)$, where χ is a smooth function that is zero if $|y| \leq 1/2$, and 1 if $|y| \geq 1$.

Example 1.8. Let $a(x,\xi) \in S^{-\infty}(\Omega)$, $\chi_j \in C_0^\infty(\mathbb{R})$ s.t. $0 \leq \chi_j \leq 1, \operatorname{supp}\chi_j \subset (j-2,j), \sum_{j=1}^\infty \chi_j(t) = 1$ for all $t \geq 0$. Then

$$a(x,\xi) = \sum_{j=1}^\infty \chi_j(|\xi|)a(x,\xi).$$

The right-hand side is a classical symbol.

Definition 1.9. We say that $p \in \mathcal{D}'(\Omega \times \Omega)$ is *properly supported* if $\operatorname{supp} p$ has compact intersection with $K \times \Omega$ and $\Omega \times K$, for any compact $K \subset \Omega$.

Definition 1.10. Let A be given by

$$Au(x) = (2\pi)^{-n}\iint a(x,y,\xi)u(y)e^{i(x-y)\cdot\xi}dyd\xi,$$

where $a \in S^m(\Omega)$ is the sense that

$$|D_y^\gamma D_x^\beta D_\xi^\alpha a(x,y,\xi)| \leq C(1+|\xi|)^{m-|\alpha|},$$

on compact subsets of $\Omega \times \Omega$, and assume that A is compactly supported. Then $A\colon C_0^\infty(\Omega) \to C^\infty(\Omega)$ and we can extend this to $A\colon C^\infty \to C^\infty(\Omega)$. In that case, we say that a is the *amplitude* of A and we write $A = \mathrm{Op}(a)$.

Proof.

- First part: If $u \in C_0^\infty(\Omega)$, then
$$\left|\int u(y)a(x,y,\xi)e^{-iy\cdot\xi}dy\right| \le C_N(1+|\xi|)^{m-N}.$$
So, this is absolutely integrable.
- Second part: Let $u \in C^\infty(\Omega)$. A prop supp $\implies \exists v \in C_0^\infty(\Omega)$ s.t. $v = 1$ on a neighborhood of $\tilde{K}$, where $\tilde{K}\times\tilde{K} \supset \operatorname{supp} A \cap (\Omega \times K)$. So $A(u) = A(vu) \in C^\infty$ by previous part.

□

Example 1.11. Let $X_w f(z,\theta) = \int_{\mathbb{R}} w(z+t\theta,\theta)f(z+t\theta)dt$ be the weighted X-ray transform. Then the normal operator $N_{b,c} = X_b^* X_c$ has Schwartz kernel $K(x,y,x-y) = W(x,y,\frac{x-y}{|x-y|})/|x-y|^{n-1}$, where
$$W(x,y,\theta) = \overline{b}(x,\theta)c(y,\theta) + \overline{b}(x,-\theta)c(y,-\theta).$$
Then, $N_{b,c} = \operatorname{Op}(a)$, where a is the Fourier transform of K on the third variable. Formally:
$$\begin{aligned} a(x,y,\xi) &= \int_{\mathbb{R}^n} e^{-iz\cdot\xi}K(x,y,z)dz \\ &= \int_{\mathbb{R}^n} e^{-iz\cdot\xi}\frac{W(x,y,z/|z|)}{|z|^{n-1}}dz \\ &= \int_{\mathbb{R}_+\times S^{n-1}} e^{-ir\theta\cdot\xi}W(x,y,\theta)drd\theta \\ &= \pi\int_{S^{n-1}} W(x,y,\theta)\delta(\theta\cdot\xi)d\theta. \end{aligned}$$
The delta means that we integrate over θ's perpendicular to ξ. For example, in dimension 2, the integral reduces to
$$\frac{\pi(W(x,y,\xi_\perp/|\xi|) + W(x,y,-\xi_\perp/|\xi|))}{|\xi|}.$$

Theorem 1.12. *Let $A = \operatorname{Op}(a) \in \Psi^m$ be properly supported. Then, there is $p \in S^m$ such that $A = \operatorname{Op}(p)$. Furthermore, we have $p(x,\xi) = e^{-ix\cdot\xi}A(e^{ix\cdot\xi})$, and we have the asymptotic expansion*
$$p(x,\xi) \sim \sum_{\alpha\ge 0}\frac{i^{|\alpha|}}{\alpha!}D_\xi^\alpha D_y^\alpha a(x,y,\xi)\bigg|_{y=x}.$$

Proof.

- That A is prop. supp. implies
$$p(x,\xi) = e^{-ix\cdot\xi}(2\pi)^{-n}\iint a(x,y,\eta)e^{iy\cdot\xi}e^{i(x-y)\cdot\eta}dyd\eta$$
is well defined. Furthermore
$$\begin{aligned} Au(x) &= (2\pi)^{-n}\iint a(x,y,\eta)\left(\int \hat{u}(\xi)e^{iy\cdot\xi}d\xi\right)e^{i(x-y)\cdot\eta}dyd\eta \\ &= (2\pi)^{-n}\int \hat{u}(\xi)\left(\iint a(x,y,\eta)e^{iy\cdot\xi}e^{i(x-y)\cdot\eta}dyd\eta e^{-ix\cdot\xi}\right)e^{ix\cdot\xi}d\xi \end{aligned}$$

$$= (2\pi)^{-n} \int \hat{u}(\xi) p(x,\xi) d\xi = \mathrm{Op}(p).$$

- For the expansion, formally we have

$$e^{iD_\xi \cdot D_y} a(x,y,\xi) = (2\pi)^{-2n} \iiiint e^{i((y-y')\cdot\eta + (\xi-\xi')\cdot z)} e^{iz\cdot\eta} a(x,y',\xi') dy' d\eta d\xi' dz,$$

and

$$\int e^{i((y-y')\cdot\eta + (\xi-\xi')\cdot z)} e^{iz\cdot\eta} dz = (2\pi)^n \delta(\eta + (\xi - \xi')) e^{i(y-y')\cdot\eta}.$$

So,

$$\begin{aligned} e^{iD_\xi \cdot D_y} a(x,y,\xi) &= (2\pi)^{-n} \iiint \delta(\eta + (\xi-\xi')) e^{i(y-y')\cdot eta} a(x,y',\xi') dy d\xi' d\eta \\ &= (2\pi)^{-n} \iint e^{i(y-y')\cdot(\xi'-\xi)} a(x,y',\xi') dy' d\xi' \end{aligned}$$

Put $y = x$ and use the formal expansion of $e^{D_\xi \cdot D_y}$ to obtain

$$p(x,\xi) = e^{iD_\xi \cdot D_y} a(x,y,\xi)|_{y=x} = \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha a(x,y,\xi) \bigg|_{y=x}$$

□

Proposition 1.13. *If $P \in \Psi^m$ is properly supported, then $P', P^* \in \Psi^m$.*

Proof. We have

$$\begin{aligned} (Pu,v) &= (2\pi)^{-n} \int v(y) \iint e^{i(y-x)\cdot\xi} p(y,\xi) v(x) dx d\xi dy \\ &= (2\pi)^{-n} \int u(x) \iint e^{i(x-y)\cdot\xi} p(y,-\xi) v(x) dy d\xi dx = (u, p(x,-D)v). \end{aligned}$$

So $P'v(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} p(y,-\xi) v(y) dy d\xi$. Similarly,

$$\begin{aligned} (Pu,\overline{v}) &= (2\pi)^{-n} \int \overline{v}(y) \iint e^{i(y-x)\cdot\xi} p(y,\xi) u(x) dx d\xi dy \\ &= (2\pi)^{-n} \int \overline{u}(x) \overline{\iint e^{i(x-y)\cdot\xi} \overline{p}(y,\xi) v(y) dy d\xi} dx = (u, \overline{\overline{p(x,D)}v}). \end{aligned}$$

So $P^* v(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} \overline{p}(y,\xi) v(y) dy d\xi$. □

Furthermore, we conclude that

Corollary 1.14. *If $P = \mathrm{Op}(p) \in \Psi^m$ is properly supported, then*

$$\sigma(P')(x,\xi) \sim \sum_{\alpha \geq 0} \frac{(-1)^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha p(x,-\xi), \quad \sigma(P^*)(x,\xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \overline{p}(x,\xi).$$

If $A = \mathrm{Op}(a)$ is properly supported, then the amplitude of A' and A^* are

$$a(y,x,-\xi), \quad \overline{a}(y,x,\xi).$$

Proposition 1.15. *Let $P = \mathrm{Op}(p) \in \Psi^m$, $Q = \mathrm{Op}(q) \in \Psi^\mu$ with Q properly supported. Then, $PQ \in \Psi^{m+\mu}$. Furthermore, $p(x,D)q(x,D) = r(x,D)$ with*

$$r(x,\xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi).$$

Proof. Since Q is properly supported and $Q = Q^{**}$, we obtain

$$\begin{aligned} Qu(x) &= Q^{**}u(x) \\ &= (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} q(y,\xi)u(y)dyd\xi \\ &= (2\pi)^{-n}\mathcal{F}^{-1}\left(\int q(y,\xi)e^{-iy\cdot\xi}u(y)dy\right)(x). \end{aligned}$$

So,

$$\widehat{Qu}(\xi) = (2\pi)^{-n}\int q(y,\xi)e^{-iy\cdot\xi}u(y)dy.$$

Then,

$$PQu(x) = \int e^{ix\cdot\xi}p(x,\xi)\widehat{Qu}(\xi)d\xi = (2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}p(x,\xi)q(y,\xi)u(y)dyd\xi.$$

Hence, $PQu = \operatorname{Op}(a)$ with $a(x,y,\xi) = p(x,\xi)q(y,\xi)$. Finally,

$$\begin{aligned} r(x,y) &\sim \sum_{\alpha\geq 0}\frac{i^{|\alpha|}}{\alpha!}D_\xi^\alpha D_y^\alpha(p(x,\xi)q(y,\xi))\Big|_{y=x}, \\ &= \sum_{\gamma,\sigma\geq 0}\frac{i^{|\sigma|-|\gamma|}}{\sigma!\gamma!}D_\xi^\sigma D_y^\sigma(p(x,\xi)D_\xi^\gamma D_x^\gamma q(y,\xi))\Big|_{y=x} \\ &= \sum_{\alpha}\frac{i^{|\alpha|}}{\alpha!}D_\xi^\alpha p(x,\xi)\sum_{\beta,\gamma}\frac{i^{|\beta|-|\gamma|}}{\beta!\gamma!}D_\xi^{\beta+\gamma}D_x^{\beta+\gamma+\alpha}q(x,\xi)\Big|_{y=x} \\ &= \sum_{\alpha\geq 0}\frac{i^{|\alpha|}}{\alpha!}D_\xi^\alpha p(x,\xi)D_x^\alpha q(x,\xi). \end{aligned}$$

□

Remark 1.16. One can compose two ΨDOs (not necessarily properly supported) by the price of adding an smoothing operator to Q. Indeed, let K be the Schwartz kernel of Q, and $\chi = 1$ on the diagonal. Then, the ΨDO with Schwartz kernel $(1-\chi)K$ is smooth.

Lemma 1.17. *Let $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ be open sets and $\chi\colon \Omega \to \tilde{\Omega}$ be a diffeomorphism. Assume $P = p(x,D) \in \Psi^m(\Omega)$ is properly supported. Define $\tilde{P}u := P(u\circ\chi)\circ\chi^{-1}$. Then $\tilde{P} \in \Psi^m(\tilde{\Omega})$ and $\tilde{P} = \tilde{p}(x,D)$, where*

$$\tilde{p}(\chi(x),\xi) \sim \sum_{\alpha\geq 0}\frac{1}{\alpha!}\varphi_\alpha(x,\xi)D_\xi^\alpha(x, J\xi^t(x)\xi),$$

where $\varphi_\alpha(x,\xi)$ is a polynomial in ξ of degree $\leq |\alpha|/2$, with $\varphi_0(x,\xi) = 1$.

Proof. Let $\theta := \chi^{-1}$. Using a changes of variables we see that

$$\begin{aligned} \tilde{P}u(x) &= (2\pi)^{-n}\iint e^{i(\theta(x)-y)\cdot\xi}p(\theta(x),\xi)u(\chi(y))dyd\xi \\ &= (2\pi)^{-n}\iint e^{i(\theta(x)-\theta(y))\cdot\xi}p(\theta(x),\xi)u(y)\det J\theta(y)dyd\xi \\ &= (2\pi)^{-n}\iint e^{i(x-y)\cdot\Phi^t(x,y)\xi}p(\theta(x),\xi)\det J\theta u(y)dyd\xi, \end{aligned}$$

where Φ satisfies

$$(\theta(x)-\theta(y))\cdot\xi=\sum_{j=1}^{n}(\theta^j(x)-\theta^j(y))\xi_j=\sum_{j,k}\Phi_{k,j}(x,y)(x_k-y_k)\xi_j.$$

Hence, Φ is smooth near the diagonal in $\tilde{\Omega}\times\tilde{\Omega}$ and we have

$$\tilde{P}u(x)=(2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}p(\theta(x),\psi(x,y)\xi)D(x,y)u(y)dyd\xi+Ku,$$

where $\psi(x,y)=\Phi(x,y)^{-1}$, $D(x,y)=\det J\theta(y)\det\psi(x,y)\Xi(x,y)$, where $\Xi(x,y)=1$ in a neighborhood of the diagonal in $\tilde{\Omega}\times\tilde{\Omega}$ and we put it there because $\psi(x,y)$ might no be defined everywhere. Finally, $K\in\Psi^{-\infty}$. Hence, $\tilde{P}=\mathrm{Op}(a)+K$, where $a(x,y,\xi)=p(\theta(x),\psi(x,y)\xi)D(x,y)$. Also, the chain rule shows that

$$|D_y^\gamma D_x^\beta D_\xi^\alpha a(x,y,\xi)|\le C(1+|\xi|)^{m-|\alpha|}.$$

Therefore, since $K\in\Psi^{-\infty}$, we conclude that $\tilde{P}\in\Psi^m$.

For the part of the asymptotic part, we apply Theorem 1.12 to obtain

$$\tilde{p}(x,\xi)\sim\sum_{\alpha\ge 0}\frac{i^{|\alpha|}}{\alpha!}D_\xi^\alpha D_y^\alpha(p(\theta(x),\psi(x,y)\xi)D(x,y))\Big|_{y=x}.$$

Finally, $\psi(x,x)=(J\chi)^t(\theta(x))$ and $D(x,x)=1$, so we obtain the result. □

Definition 1.18. If $P=p(x,D)\in\Psi^m$, we define its *principal symbol* to be the equivalence class $p(x,D)+S^m/S^{m-1}$. We write $\sigma(P)$ and $\sigma_{\mathrm{pr}}(P)$ to denote the total symbol and principal symbol of P, respectively.

Remark 1.19. Since

$$\tilde{p}(x,\xi)-p(\theta(x),(J\chi)^t(\theta(x))\xi)\in S^{m-1},$$

the principal symbol of $P=\mathrm{Op}(p)\in\Psi^m$ is well defined on manifolds. This is not true for the whole symbol, which can be easily seen by taken, for example, the Laplace–Beltrami operator and looking its transformation by a change of coordinates.

Remark 1.20. For classical ΨDOs we have

$$\sigma_{\mathrm{pr}}(P)(x,\xi)=\lim_{\lambda\to\infty}\lambda^{-m}e^{-i\lambda x\cdot\xi}P(e^{i\lambda x\cdot\xi}).$$

At a first glace, this definition looks mysterious, so let us explore what it is mean.

Example 1.21.

(1) If $P=\sum_{|\alpha|\le k}a_\alpha(x)D^\alpha$. Then $\sigma(P)(x,\xi)=\sum_{|\alpha|\le k}a_\alpha(x)\xi^\alpha$ and $\sigma_{\mathrm{pr}}(P)=\sum_{|\alpha|=k}a_\alpha(x)\xi^\alpha$.

(2) In a more general ways, classical symbols (see Definition 1.6), i.e., symbols of the form $p\sim\sum_{j=0}^\infty p_{m-j}$, the principal symbol is p_m.

(3) For properly supported operators $\mathrm{Op}(a)$, we have that the principal symbol is $a(x,x,\xi)$. Indeed, recall that from Theorem 1.12, we have that $\mathrm{Op}(a)=\mathrm{Op}(p)$ with $p\sim\sum_{|\alpha|\ge 0}D_\xi^\alpha D_y^\alpha a(x,y,\xi)|_{y=x}$. From the previous point, we conclude that the principal symbol of $\mathrm{Op}(p)$ is $a(x,x,\xi)$ as we claimed at the beginning. In particular, we have that

$$\sigma_p(N_{b,c})=\pi\int_{S^{n-1}}W(x,x,\theta)\delta(\theta\cdot\xi)=2\pi\int_{S^{n-1}}\bar{b}(x,\theta)a(x,\theta)\delta(\theta\cdot\xi)d\theta.$$

(4) Let $P = p(x, D) \in \Psi$ properly supported. From Corollary 1.14 we see $\sigma(P^*) \sim \sum_{|\alpha|\geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \overline{p}(x,\xi)$. So, $\sigma_{\text{pr}}(P^*) = \overline{\sigma_p(P)}$. Similarly, $\sigma_p(P')(x,\xi) = \sigma_{\text{pr}}(P)(x,-\xi)$.

(5) Let $P = \text{Op}(p) \in \Psi^m$, and $Q = \text{Op}(p) \in \Psi^\mu$ properly supported. Then $PQ \in \Psi^{m+\mu}$ has a symbol $\sim \sum_{\alpha\geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi)$. So, $\sigma_{\text{pr}}(PQ) = \sigma_{\text{pr}}(P)\sigma_{\text{pr}}(Q)$.

(6) Let $P = \text{Op}(p) \in \Psi^m$, $Q = \text{Op}(q) \in \Psi^\mu$ be properly supported. Then $PQ, QP \in \Psi^{m+\mu}$, and $[P, Q] \in \Psi^{m+\mu}$. So,

$$\begin{aligned}\sigma_{\text{pr}}([P,Q]) &= i(\nabla_\xi \sigma_{\text{pr}}(P) \cdot \nabla_x \sigma_{\text{pr}}(Q) - \nabla_\xi \sigma_{\text{pr}}(Q) \cdot \nabla_x \sigma_{\text{pr}}(P)) \\ &= \frac{1}{i} H_{\sigma_{\text{pr}}(P)} \sigma_{\text{pr}}(Q),\end{aligned}$$

where $H_{\sigma_{\text{pr}}(P)}$ is the *Hamiltonian vector field* of $\sigma_{\text{pr}}(P)$.

Lemma 1.22. *If $P \in \Psi^m(\Omega)$, then its Schwartz kernel is smooth.*

Proof. Since $\sigma(P) = p \in S^{-\infty}$, we can differentiate K (w.r.t x_j or y_j) using the dominate convergence theorem, and the integral will be always absolutely convergent. □

Corollary 1.23. *If $P \in \Psi^m(\Omega)$, then P is a compact operator.*

Proof. Since K is smooth, then P is a Hilbert–Schmidt operator with $\|P\|_{\text{HS}} = \|K\|$. Since Hilbert–Schmidt operators are compact, we are done. □

1.2. Sobolev continuity.

Lemma 1.24. *If $p(x,\xi) \in S^0(\Omega)$, and if $\text{Re}\, p(x,\xi) \geq C > 0$, then there exists a $B \in \Psi^0$ such that, with $\text{Re}\, P = (1/2)(P + P^*)$,*

$$\text{Re}\, P - B^*B \in \Psi^{-\infty}.$$

Proof.

- We want $b(x,\xi) \sim \sum b_j(x,\xi)$ with $b_j \in S^{-j}$.
- Take $b_0(x,\xi) := (\text{Re}\, p(x,\xi))^{1/2}$. Since $\text{Re}\, p \geq c$, then $b_0 \in S^0$. Also,
$$\text{Re}\, p - \overline{b}_0 b_0 = r_1 \in S^{-1}.$$
- Suppose we have the terms $b_0, \ldots, b_j$. We need $b_{j+1} \in S^{-(j+1)}$ s.t.
$$\text{Re}\, p = ((\overline{b}_0 + \cdots + \overline{b}_j) + \overline{b}_{j+1})((b_0 + \cdots + b_j) + b_{j+1}) + r_{j+1},$$
with $r_{j+1} \in S^{-(j+1)}$. By induction hypothesis, the right-hand side is equal to
$$\begin{aligned}\text{Re}\, p + r_j &+ \overline{b}_{j+1}(b_0 + \cdots + b_{j+1}) + (\overline{b}_0 + \cdots + \overline{b}_{j+1})b_{j+1} + r_{j+1} \\ &= \text{Re}\, p + r_j + \overline{b}_{j+1} b_0 + \overline{b}_0 b_{j+1} \bmod S^{-(j+1)}\end{aligned}$$
- Note that $r_j = \overline{r}_j$. So, we require
$$\overline{b}_{j+1} b_0 + \overline{b}_0 b_{j+1} = -r_j,$$
and we can take $b_{j+1} = -r_j/(2b_0)$.

□

Lemma 1.25. *Let $A \in \Psi^0(\Omega)$ with*

$$\limsup_{|\xi|\to\infty} |\sigma(A)(x,\xi)| < M < \infty.$$

If $K \subset\subset \Omega$, then $\exists R \in \Psi^{-\infty}$ such that

$$\|Au\|^2_{L^2(K)} \leq M^2\|u\|^2 + (Ru, u).$$

Proof.

- The operator $C = M^2 - A^*A$ has principal symbol $C(x,\xi) = M^2 - |\sigma_{pr}(A)(x,\xi)|^2 > 0$, so by Lemma 1.24 $\exists B \in \Psi^0$ such that

$$C - B^*B = M^2 - A^*A - B^*B = -R \in \Psi^{-\infty}.$$

  So, $A^*A + B^*B = M^2 + R$.
- Using the previous inequality and the hypothesis of A we find

$$\|A\|^2_{L^2} \leq (Au, Au) + (Bu, Bu) \leq M^2\|u\|^2_{L^2} + (Ru, u).$$

□

Theorem 1.26 (Sobolev continuity)**.** *If $P \in \Psi^m(\Omega)$ is properly supported, then $P\colon H^s_{\text{loc}}(\Omega) \to H^{s-m}_{\text{loc}}(\Omega)$ is continuous.*

Proof. Let $\Lambda^s(u) = \int (1+|\xi|^2)^{s/2} e^{ix\cdot\xi}\hat{u}(\xi)d\xi$. Note that $H^s = \Lambda^{-s}L^2$ continuously. Now, observe that $\Lambda^{-s} \in \Psi^{-s}$. So, $A\Lambda^{-s} \in \Psi^{m-s}$, and therefore, $\Lambda^{s-m}A\Lambda^{-s} \in \Psi^0$. Now, by Lemma 1.25 $\Lambda^{s-m}A\Lambda^{-s}\colon L^2_{\text{loc}} \to L^2_{\text{loc}}$ continuously (the hypothesis of the lemma are satisfied because we are working on compact sets with an operator of order 0). Thus, $A\colon H^s_{\text{loc}} \to H^{s-m}_{\text{loc}}$ continuously. □

1.3. **Elliptic regularity.**

Definition 1.27. The operator $P \in \Psi^m$ is *elliptic* of order m if on each compact $K \subset \Omega$ there are constants C_K and R

$$|\sigma(P)(x,\xi)| \geq C_K(1+|\xi|)^m, \quad x \in K, |\xi| \geq R.$$

Remark 1.28.

(1) We can always replace R by $\tilde{R} = \max\{1, R\}$ to require $|p(x,\xi)| \geq C_K|\xi|^m$ for $|\xi| \geq \tilde{R}$.
(2) If P is elliptic, then $|\sigma_{pr}(P)(x,\xi)| \geq C_K(1+|\xi|)^m - C_1(1+|\xi|)^{m-1} \geq C_2(1+|\xi|)^m$, where we used that $\sigma(P) - \sigma_{pr}(P) \in S^{m-1}$. Similarly, if $|\sigma_{pr}(P)| \geq C(1+|\xi|)^m$, then

$$C(1+|\xi|)^m \geq |\sigma_{pr}(P)| \leq |\sigma(P)| + |\sigma(P) - \sigma_{pr}(P)| \leq |\sigma(P)| + C(1+|\xi|)^{m-1}.$$

Therefore, to show that $P \in \Psi^m$ is elliptic, is enough to check that

$$|\sigma_{pr}(P)(x,\xi)| \geq C_K(1+|\xi|)^m, \quad x \in K, |\xi| \geq R.$$

Example 1.29.

(1) We have that $\sigma_p(-\Delta)(x,\xi) = |\xi|^2$. So, the Laplace operator is an elliptic operator
(2) Now, consider the wave operator $\Box = \partial_{tt} - \Delta$. Then, $\sigma_p(\Box)(t,x,\tau,\xi) = -\tau^2 + |\xi|^2$ and $\Box$ is not elliptic.

(3) $N_{b,c}$ is elliptic if and only if $0 \neq a(x,x,\xi) = 2\pi \int_{S^{n-1}} \overline{b}(x,\theta)a(x,\theta)\delta(\theta\cdot\xi)d\theta$. In particular, $N_{w,w}$ is elliptic

$$0 \neq 2\pi \int_{S^{n-1}} |w(x,\theta)|^2 \delta(\theta \cdot \xi) d\theta,$$

that is, if for every (x,ξ) there exists $\theta \in S^{n-1}$ with $\theta \perp \xi$ so that $w(x,\theta) \neq 0$.

Definition 1.30. The ΨDO Q is a *parametrix* for $P \in \Psi^m$ if is properly supported and

$$PQ - I = K_1 \in \Psi^{-\infty}, \quad QP - I = K_2 \in \Psi^{-\infty}.$$

Example 1.31. From our previous examples, we have that $Q = q(x,\xi) = \frac{1-\varphi(\xi)}{|\xi|^2}$ is a parametrix for Δ.

The key point of having an elliptic operator is that they have parametrix:

Theorem 1.32. *If $P = \mathrm{Op}(p) \in \Psi^m$ is elliptic, then there exists a properly supported $Q \in \Psi^{-m}$ which is a parametrix for P.*

Proof.

- Let
$$\zeta(x,\xi) = \begin{cases} 0 & \text{in a neighborhood of the zeros of } p, \\ 1 & \text{for } |\xi| \geq C, \end{cases}$$
and $q_0(x,\xi) := \zeta(x,\xi)p(x,\xi)^{-1} \in S^{-m}$. Let $Q_0 = q_0(x,D)$.
- Then, $\sigma(Q_0P) \sim 1 + r(x,\xi)$. So, $Q_0P = I + R$, where $R \in \Psi^{-1}$. Let $E \in \Psi^0$ be such that $\sigma(E) \sim \sum_{j=0}^{\infty} r^j$. So, $(EQ_0)P = I + K_2$, where $K_2 \in \Psi^{-\infty}$. The construction for right parametrix is similar.
- We have
$$QP\tilde{Q} = (I + K_2)\tilde{Q} = \tilde{Q} + K_2\tilde{Q}$$
$$QP\tilde{Q} = Q(I + K_1) = Q + QK_1.$$
Therefore, $Q - \tilde{Q} = K_2\tilde{Q} - QK_1 \in \Psi^{-\infty}$.

□

Definition 1.33. $x_0 \notin \operatorname{sing\,supp}(u)$ if φu is a smooth function for any smooth function φ compactly supported on neighborhood of x_0

Example 1.34. Consider $u = \delta_0$. Note that $\operatorname{supp} u = \{0\}$. So, $\operatorname{sing\,supp} u \subset \{0\}$. Suppose that $0 \notin \operatorname{sing\,supp} u$. Then, $\varphi u = \phi$ is a smooth function in a neighborhood U of 0. Take $\psi \in C_0^\infty(U)$ with $\psi(0) \neq 0$. Then,

$$\varphi(0)\psi_1(0) = \varphi\delta(\psi_1) = \int_{\mathbb{R}^n} \phi(x)\psi_1(x)dx = 0,$$

because ϕ is zero almost everywhere. This contradiction shows $0 \in \operatorname{sing\,supp} \delta_0$.

Theorem 1.35 (Pseudolocal property)**.** *If $P \in \Psi^m(\Omega)$ and $u \in \mathcal{E}'(\Omega)$, then*

$$\operatorname{sing\,supp} Pu \subset \operatorname{sing\,supp} u.$$

Proof.

- Let K be the Schwartz kernel of P. Then,
$$K(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} p(x,\xi) d\xi.$$
So
$$(x-y)^\alpha K(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} D^\alpha_\xi p(x,\xi) d\xi.$$
Since $p \in S^m$, this is absolutely convergent for α satisfying $m - |\alpha| < -n$, and C^j for $m - |\alpha| < -n - j$. Therefore, K is smooth off the diagonal.
- Let $\chi \in C_0^\infty$ be smooth on a neighborhood of $x_0 \notin \operatorname{sing\,supp}$. By the mapping properties of P, we have that $K(\chi u)(x)$ is smooth. Let χ_1 be 1 on an smaller neighborhood. Then, $\operatorname{supp} \chi_1(x)[K(1-\chi)u](x)$ does not intersect the diagonal, and hence this function is smooth. By adding both terms, we conclude that Ku is smooth on x_0.

□

Theorem 1.36 (Elliptic regularity)**.** *If Ψ^m is a properly supported elliptic operator, then for any $u \in \mathcal{D}'(\Omega)$ we have*
$$\operatorname{sing\,supp} Pu = \operatorname{sing\,supp} u.$$

Proof.

- By the Pseudolocal property $\operatorname{sing\,supp} Pu \subset \operatorname{sing\,supp} u$.
- $\exists Q \in \Psi^{-m}$ s.t. $u = QPu \bmod C^\infty$. This and the Pseudolocal property imply $\operatorname{sing\,supp} u = \operatorname{sing\,supp} Q(Pu) \subset \operatorname{sing\,supp} Pu$.

□

1.4. **Wafe front sets.**

Definition 1.37. Let $P \in \Psi^m$ with homogeneous principal symbol homogeneous in ξ) of degree m. The *characteristic set* of P is
$$\operatorname{char} P = \{(x,\xi) \in T^*\Omega \setminus 0 : \sigma_{\text{pr}}(P)(x,\xi) = 0\}.$$

Example 1.38. Consider the operator P with symbol is $p(x,\xi) = \sum_{|\alpha|\le k} a_\alpha(x)\xi^\alpha$. Recall that the principal symbol is given by $\sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$. Hence, the operator is elliptic if and only if $\operatorname{char} P = \emptyset$.

Definition 1.39. Let $u \in \mathcal{D}'(\Omega)$. We define its *wave front set* by
$$\operatorname{WF}(u) = \bigcap_{\substack{P\in\Psi^0 \\ Pu \in C^\infty}} \operatorname{char} P.$$

The moral here is that the wave front set does not only focus on the points in which u is singular, but only about the directions in which u is singular. So, the wave front set is a generalization of the singular support. More specifically, we have the following result.

Theorem 1.40. *Let $u \in \mathcal{D}'(\Omega)$, $\pi \colon T^*\Omega \to \Omega$. Then,*
$$\pi(\operatorname{WF}(u)) = \operatorname{sing\,supp} u.$$

Proof. Work using contrapositive:

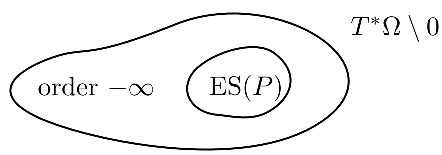
- If $x_0 \notin \operatorname{sing\,supp} u \Longrightarrow \exists \varphi \in C_0^\infty(\Omega)$, with $\varphi = 1$ near x_0 s.t. $\varphi u \in C_0^\infty(\Omega)$. Now $(x_0, \xi) \notin \operatorname{char} \varphi$ for any $\xi \neq 0 \Longrightarrow x_0 \notin \pi(\mathrm{WF}(u))$.
- If $x_0 \notin \pi(\mathrm{WF}(u))$, given $|\xi| = 1$ $\exists Q \in \Psi^0$ s.t. $(x_0, \xi) \notin \operatorname{char} Q$ and $Qu \in C^\infty$. Compacteness $\Longrightarrow \forall (x_0, \xi)$, $\exists Q_j$ s.t $(x_0, \xi) \notin \operatorname{char} Q_j$. $Q := \sum Q_j^* Q_j \in \Psi^0$ is elliptic near x_0 and $Qu \in C^\infty$. E.R. $\Longrightarrow u \in C^\infty$ near x_0.

□

Definition 1.41. Let $U \subset T^*\Omega \setminus 0$ be conic and open. $p \in S^m$ has order $-\infty$ on U if for each closed conic subset $K \subset U$ with $\pi(K)$ compact, we have

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{\alpha,\beta,N,K}(1 + |\xi|)^{-N}, \quad (x, \xi) \in K, \forall N.$$

Definition 1.42. If $P = p(x, D) \in \Psi^m$, we define the *essential support* (sometimes called *microsupport*) of P (and of p), and we write $\mathrm{ES}(P) = \mathrm{ES}(p)$, to be the smallest closed conic subset of $T^*\Omega \setminus 0$ on the complement of which p has order $-\infty$.



Proposition 1.43. $(x_0, \xi_0) \notin \mathrm{WF}(u) \iff \exists \varphi \in C_0^\infty(\Omega)$ *with* $\varphi(x_0) \neq 0$, *and a conic neighborhood* Γ *of* ξ_0 *such that, for every* N,

$$|\widehat{\varphi u}(\xi)| \leq C_N(1 + |\xi|)^{-N}, \quad \xi \in \Gamma.$$

Proof.

- Assume that the new condition holds and let $\chi(\xi)$ satifying:
  - homogeneous of degree 0 in ξ, for $|\xi| \geq c_0 > 0$,
  - $\chi(\xi_0) \neq 0$,
  - $\operatorname{supp} \chi \subset \Gamma$.

  Then, $\chi(D)\varphi u \in C^\infty$, $\chi(D)\varphi \in \Psi^0$ and $(x_0, \xi_0) \notin \operatorname{char}(\chi(D)\varphi)$, so, $(x_0, \xi_0) \notin \mathrm{WF}(u)$.
- Now let $(x_0, \xi_0) \notin \mathrm{WF}(u)$. We can construct $\varphi(x)$ and $\chi(\xi)$ with $\varphi(x_0) \neq 0$, $\chi(\xi_0) \neq 0$, and s.t. $\mathrm{ES}(\chi(D)\varphi) \cap \mathrm{WF}(u) = \emptyset$. Regularizing χ, one can suppose that $\hat{\chi}$ has compact support. Thus $\chi(D)\varphi u \in C_0^\infty$, so $\chi(\xi)\widehat{\varphi u} = O((1 + |\xi|)^{-N})$ for every N.

□

Remark 1.44. This characterization can also be stated as follows: $(x_0, \xi_0) \notin \mathrm{WF}(u) \iff \exists P = p \in \Psi^0$ with $\sigma_{\mathrm{pr}}(P)(x_0, \xi_0) \neq 0$ and $Pu \in C^\infty$.

Besides the definition of wave front set is useful to work abstractly, the characterization by decay of Fourier transform is useful to think of examples.

Example 1.45.

(1) Take $u = \delta_0 \in \mathcal{D}'(\mathbb{R}^n)$. Let U a neighborhood of 0, $\varphi \in C_0^\infty(U)$. Then,

$$\widehat{\varphi u}(\xi) = (\varphi u)(e^{-i\langle \cdot, \xi_0 \rangle}) = u(\varphi e^{-i\langle \cdot, \xi_0 \rangle}) = \varphi(0).$$

Since we have no decay in any direction, we conclude $\mathrm{WF}(u) = \{(0,\xi) : \xi \in \mathbb{R}^n \setminus 0\}$. Note that this and Example 1.34 show that Theorem 1.40 is satisfied on the case of $u = \delta_0$.

(2) Write $x = (x', x'')$, where $x' = (x_1, \dots, x_k)$, $x'' = (x_{k+1}, \dots, x_n)$. Define

$$\delta(x')(\varphi) = \int \varphi(0, x'')dx''.$$

Take $u = \delta(x')$. Let U a neighborhood of $x_0 = (x_0', x_0'')$, $\varphi \in C_0^\infty(U)$. Then,

$$\widehat{\varphi u}(\xi_0) = \varphi u(e^{-i\langle \cdot, \xi_0\rangle}) = u(\varphi e^{-i\langle \cdot, \xi_0\rangle}) = \int \varphi(0, x'')e^{-i\langle (0,x''), \xi_0\rangle}dx''.$$

The integral on the right-hand side is just the Fourier transform of $\varphi_0(y) := \varphi(0, y)$ at ξ_0''. Hence, is rapidly decreasing if $\xi_0'' \neq 0$. Therefore,

$$\mathrm{WF}(u) \subset \{(x,\xi) : x = (0, x''),\ \xi = (\xi', 0) \text{ with } \xi' \neq 0\}.$$

The other contention is clear. So, $\mathrm{WF}(\delta(x')) = N^*\{x' = 0\}$.

Lemma 1.46. $\mathrm{ES}(PQ) \subset \mathrm{ES}(P) \cap \mathrm{ES}(Q)$.

This follows from the formula for the symbol of the composition of ΨDOs.

Lemma 1.47. *Let $u \in \mathcal{D}'(\Omega)$ and suppose U is a conic open subset of $T^*\Omega \setminus 0$ with $U \cap \mathrm{WF}(u) = \emptyset$. If $P \in \Psi^m$ with $\mathrm{ES}(P) \subset U$, then $Pu \in C^\infty$.*

Proof.

- Let $P_0 \in \Psi^0$ with symbol $\equiv 1$ on a conic neighborhood V of $\mathrm{ES}(P)$. Then,

$$P(1 - P_0) = \begin{cases} 0 & \text{in } V, \\ \text{has order } -\infty & \text{in } V^c. \end{cases}$$

  In particular $P(1 - P_0) \in \Psi^{-\infty}$. Hence, is enough to show that $P_0u \in C^\infty$. WLOG $P \in \Psi^0$.
- $\mathrm{ES}(P) \cap \mathrm{WF}(u) = \emptyset$ and $T^*\Omega$ is second countable $\implies \exists Q_j \in \Psi^0$ s.t. $Q_ju \in C^\infty$ and each $(x, \xi) \in \mathrm{ES}(P)$ is noncharacteristic for some Q_j. So, if $Q = \sum Q_j^*Q_j$, then $Qu \in C^\infty$ and char $Q \cap \mathrm{ES}(P) = \emptyset$.
- Let $\tilde{Q}$ be elliptic with $\sigma(\tilde{Q}) = \sigma(Q)$ on a conic neighborhood of $\mathrm{ES}(P)$, and let $\tilde{Q}^{-1}$ denote a parametrix for $\tilde{Q}$. Set $A = P\tilde{Q}^{-1}$. Then

$$AQ = P\tilde{Q}^{-1}(\tilde{Q} + Q - \tilde{Q}) = P + P\tilde{Q}^{-1}(Q - \tilde{Q}) = P \bmod \Psi^{-\infty}.$$

  Therefore $Pu = AQu \in C^\infty$ up to a smooth function.

□

Theorem 1.48 (Microlocal pseudolocal property)**.** *Let $P \in \Psi^m$. Then*

$$\mathrm{WF}(Pu) \subset \mathrm{WF}(u) \cap \mathrm{ES}(P).$$

Proof.

- Take $(x_0, \xi_0) \notin \mathrm{ES}(P)$. Let $Q = q(x, D) \in \Psi^0$ so that $q(x, \xi) = 1$ on a conic neighborhood of (x_0, ξ_0) and $\mathrm{ES}(Q) \cap \mathrm{ES}(P) = \emptyset$. Then $QP \in \Psi^{-\infty}$, so $QPu \in C^\infty$. Since $q(x, \xi) = 1$ on a neighborhood of (x_0, ξ_0), then $(x_0, \xi_0) \notin \mathrm{WF}(Pu)$.

- Let Γ be any conic neighborhood of $\mathrm{WF}(u)$ and write $P = P_1 + P_2$, where $P_j \in \Psi^m$ satisfy $\mathrm{ES}(P_1) \subset \Gamma$ and $\mathrm{ES}(P_2) \cap \mathrm{WF}(u) = \emptyset$. Lemma 1.47 $\Longrightarrow P_2 u \in C^\infty$. Hence, $\mathrm{WF}(Pu) = \mathrm{WF}(P_1 u) \subset \Gamma$. Since this is valid for any conic neighborhood Γ of $\mathrm{WF}(u)$, we conclude that $\mathrm{WF}(Pu) \subset WF(u)$.

□

Theorem 1.49 (Microlocal regularity)**.** *If $P \in \Psi^m$ is elliptic, then*

$$\mathrm{WF}(u) = \mathrm{WF}(u).$$

Proof. One part follows from the microlocal pseudolocal property. For the other one, take a parametrix $Q \in \Psi^{-m}$ for P. Then, $\mathrm{WF}(u) = \mathrm{WF}(QPu) \subset \mathrm{WF}(Pu)$.

□

1.5. **Propagation of singularities.**

Definition 1.50. Let p be a real valued function on T^*M. We define its *Hamiltonian vector field* by

$$H_p = \sum_{j=1}^{n} \left(\frac{\partial p}{\partial x_j} \frac{\partial}{\partial \xi_j} - \frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial x_j} \right).$$

Remark 1.51*.* Note that $H_p p = 0$. So, p is constant over characteristic curves of H_p.

Definition 1.52. Let $P \in S^m$ with $\sigma_{\mathrm{pr}}(P) = p$.

- The integral curves of H_p are called the *bicharacteristics* of p. Explicitly, they satisfy
$$\dot{x} = \partial_\xi p(x, \xi), \quad \dot{\xi} = -\partial_x p(x, \xi).$$
- The integral curves on which $p = 0$ are called the *null bicharacteristics* or *zero bicharacteristics*.

Example 1.53. Let $M \subset \mathbb{R}^n$ with a Riemannian metric $g = (g_{ij}(x))$. Consider the wave operator

$$P(t, x, \partial_t, \partial_x) = \partial_{tt} - \Delta_g = \partial_{tt} - \sum_{i,j=1}^{n} g^{ij}(x) \partial_{x_i x_j} + \text{l.o.t.},$$

Then,

$$\sigma_p(P)(t, x, \tau, \xi) = -\left(\tau^2 - \sum_{i,j}^{n} g^{ij}(x) \xi_i \xi_j \right) = -\tau^2 + |\xi|_g^2.$$

Hence, the bicharacteristics of $\sigma_p(P)$ satisfy

$$\begin{cases} \dot{t} &= [\sigma_p(P)]_\tau = -2\tau, \\ \dot{x}^j &= \nabla_\xi[\sigma_p(P)] = 2\sum_i g^{ij}\xi_i, \\ \dot{\tau} &= -[\sigma_p(P)]_t = 0, \\ \dot{\xi}^j &= -\nabla_x[\sigma_p(P)] = -2\partial_{x^j} \sum_{i,j} g^{ij}(x)\xi_i\xi_j. \end{cases}$$

The latter two for x and ξ are the Hamiltonian curves of $\tilde{H} := \sum g^{ij}(x)\xi_i\xi_j$ and they are known to coincide with the geodesics $(\gamma, \dot{\gamma})$ on TM when identifying vectors and covectors by the metric. The equation for τ implies that τ is constant. The equation for t says that up to rescaling, we can choose t as a parameter for the geodesics. That rescaling forces the speed along the geodesic to be 1. Since we are studying zero bicharacteristics, we have $\tau^2 = |\xi|_g^2$. This equation defines two smooth surfaces away from $(\tau, \xi) = (0, 0)$, namely $\tau = \pm|\xi|_g$. This corresponds to geodesics starting from x in direction either ξ or $-\xi$.

Example 1.54. Consider the ΨDO $P = \partial_{x_n}$, f with compact support. The rays parallel to the x_n-axis correspond to the bicharacteristics of P.

Let $P_0 \in \Psi^m$. Consider a hyperbolic equation of the form

$$\frac{\partial u}{\partial t} = iA(t, x, D_x)u,$$

where we assume $A = A_1 + A_0$ with $A_1(t, x, \xi) \in S^1_{\mathrm{cl}}$ is real, and $A_0(t, x, \xi) \in S^0_{\mathrm{cl}\,l}$. Furthermore, we suppose $A_1(t, x, \xi)$ is homogeneous in ξ, for $|\xi| \geq 1$. Denote by $S(t, s)$ the solution operator to the equation, taking $u(s)$ to $u(t)$. This is a bounded operator on each Sobolev space H^σ, with inverse $S(s, t)$. Let $P(t) = S(t, 0)P_0S(0, t)$.

Theorem 1.55 (Y. Egorov)**.** *If $P_0 = p_0(x, D) \in \Psi^m$, then for each t, $P(t) \in \Psi^m$, modulo a smoothing operator. The principal symbol of $P(t)(\mathrm{mod} S^{m-1})$ at a point (x_0, ξ_0) is equal to $p_0(y_0, \eta_0)$, where (y_0, η_0) is obtained from (x_0, ξ_0) by following the flow $\mathcal{C}(t)$ generated by the (time-dependent) Hamiltonian vector field $H_{A_1(t,x,\xi)}$.*

Proof.

- $\frac{\partial}{\partial t}S(t, 0) = iA(t, x, D_x)S(t, 0)$, $\frac{\partial}{\partial t}S(0, t) = -S(0, t)iA(t, x, D_x)$ imply
$$P'(t) = i[A(t, x, D), P(t)], \quad P(0) = P_0.$$
- We want $Q(t) \in \Psi^m$ with $\sigma(Q(t)) = q(t, x, D) \sim q_0(t, x, \xi) + q_1(t, x, \xi) + \cdots$, solving
$$Q'(t) = i[A(t, x, D), Q(t)] + R(t), \quad Q(0) = P_0, \tag{1}$$
where $R(t) \in \Psi^{-\infty}$ for every t, and $Q(t) - P(t) \in \Psi^{-\infty}$.
- Define q_0 by
$$\left(\frac{\partial}{\partial t} - H_{A_1}\right) q_0(t, x, \xi) = 0, \quad q_0(0, x, \xi) = p_0(x, \xi),$$
and inductively define q_j by
$$\left(\frac{\partial}{\partial t} - H_{A_1}\right) q_j(t, x, \xi) = b_j(t, x, \xi), \quad q_j(0, x, \xi) = 0,$$
where $b_j = \sigma_{pr}(i[A, Q_j] - Q'_j)$. Hence, $Q = \sum_{j=0}^{\infty} q_j \in S^m$ satisfies (1).
- $P(t) - Q(t) \in \Psi^{-\infty} \iff \forall f \in H^\sigma(\mathbb{R}^n)$,
$$v(t) - w(t) := S(t, 0)P_0f - Q(t)S(t, 0)f \in H^\infty(\mathbb{R}^n).$$
- Properties of the solution operator imply
$$\frac{\partial v}{\partial t} = iA(t, x, D)v, \quad v(0) = P_0f.$$
Since Q solves (1)
$$\frac{\partial w}{\partial t} = iA(t, x, D)w + g, \quad w(0) = P_0f,$$
where $g = R(t)S(t, 0)w \in C^\infty(\mathbb{R}, H^\infty(\mathbb{R}^n))$. Regularity of hyperbolic equations gives $v(t) - w(t) \in H^\infty$, for any $f \in H^\sigma(\mathbb{R}^n)$.

□

Using this corolally of Egorov's theorem, we can prove Hörmander's propagation of singularities theorem. Consider $P \in \Psi^m$ with $\sigma(P) \sim p_m(x,\xi) + p_{m-1}(x,\xi) + \cdots$ being real.

Theorem 1.56. *Assume that $Pu = f$, where $u \in \mathcal{D}'(M)$. Then*

$$\mathrm{WF}(u) \subset \mathrm{WF}(f) \cup \operatorname{char} P,$$

and $\mathrm{WF}(u) \setminus \mathrm{WF}(f)$ is invariant under $\mathcal{C}(t)$ acting on $T^*M \setminus \mathrm{WF}(f)$.

Example 1.57. In the case of the wave equation of manifolds, we know that the bicharacteristics are geodesics (see Example 1.53). Hörmander's theorem implies the following in this case. For the equation $\partial_{tt} - \Delta_g u = 0$, we get that each singularity (x,ξ) of the initial conditions at $t = 0$ starts to propagate from x in direction either ξ or $-\xi$ or both (depending on the initial conditions) along the unit speed geodesic.

Example 1.58. Regarding ∂_{x_n} (Example 1.54), Hörmander's theorem implies that when $Pu \in C^\infty$, either no singularity of u belongs to any given bicharacteristic, or all points of it are in $\mathrm{WF}(u)$. If the bicharacterics are non-trapping, i.e., they eventually leave every compact set, the second alternative is impossible. Thus, $u \in C_0^\infty$.

Proof of Hörmanders theorem via Egorov's theorem.

- WLOG $u \in \mathcal{E}'(\Omega)$.
- If $A = a(x,D) \in \Psi^{1-m}$ has principal symbol $a_{1-m}(x,\xi) = |\xi|^{1-m}$. Then,
  $$\lambda(x,D)u = A(x,D)Pu = A(x,D)f =: g,$$
  where $\lambda(x,D) \in \Psi^1$ has principal symbol $\lambda_1(x,\xi) = |\xi|^{1-m}p_m$.
- Let $v(t,x) = u(x)$. Then
  $$\frac{\partial}{\partial t}v = i\lambda(x,D)v - ig.$$
  By Duhamel's principle,
  $$v(t) = e^{it\lambda}u - i\int_0^t e^{i(t-s)\lambda} g ds.$$
- Take $(x_0,\xi_0) \notin \mathrm{WF}(u)$. Theorem 1.48 implies $(x_0,\xi_0) \notin \mathrm{WF}(f)$ and $(x_0,\xi_0) \notin \mathrm{WF}(g)$.
- Take $\mathcal{C}(t)(x_0,\xi_0) \notin \mathrm{WF}(f)$ for $t \in (t_0,t_1)$. By Egorov's theorem
  (i) $\mathrm{WF}(e^{it\lambda}u) = \mathcal{C}(t)\,\mathrm{WF}(u)$
  (ii) $\mathrm{WF}(\int_0^s e^{i\lambda t} g dt) \subset \mathcal{C}(s)\,\mathrm{WF}(g)$.
  Then, $C(t)(x_0,\xi_0) \notin \mathrm{WF}(v(t))$ for $t \in (t_0,t_1)$. Since $v(t,x) = u$ for all t, we conclude that $C(t)(x_0,\xi_0) \notin \mathrm{WF}(u)$.

□

2. Weighted X-ray transform via microlocal analysis

2.1. **Microlocal stability of the weighted X-ray transform.**

The X-ray transform is not a ΨDO, so we cannot use the Sobolev continuity directly. However, we have

Theorem 2.1. *Let $w \in C^\infty$. Then for every $s \geq 0$,*

$$X_w \colon H^{s-1/2}_{\text{comp}}(\mathbb{R}^n) \to H^s(\Sigma), \quad X_w' : H^{s-1/2}(\Sigma) \to H^s_{\text{loc}}(\mathbb{R}^n),$$

are continuous.

Proof.

- Take $f \in C_0^\infty(\Omega)$. WLOG take w supported in a small neighborhood of a fixed line ℓ_0 (and use partition of unity), and since f is compactly supported, the same holds for w. Finally, let (y,ζ) be local coordinates on $\Sigma \times S^{n-1}$.
- Note $\partial_{y^j} X_w f = X_{w_{y^j}} f + X_w \partial_{y^j} f$. So, $X_w^* \partial_{y^j} X_w \in \Psi^0$. On the other hand, it can be shown that $\partial_{\zeta^j} X_w$ has the same structure than $\partial_{y^j} X_w$. Hence, $X_w^* \partial^{2\alpha}_{y,\zeta} X_w \in \Psi^{2|\alpha|-1}$.
- Let $\Lambda = (I-\Delta)^{1/2}$ and write $f = \Lambda^{-|\alpha|+1/2}\Lambda^{|\alpha|-1/2} f$. Let $\chi \in C_0^\infty$ be such that $\chi = 1$ near $\overline{\Omega}$. Then,

$$\begin{aligned}
\|X_w f\|^2_{H^k(\Sigma)} &= \sum_{|\alpha|\leq k} \left\|\partial^\alpha_{y,\zeta} X_w f\right\|^2_{L^2(\Sigma)} \\
&= \sum_{|\alpha|\leq k} \left|(X_w^* \partial^{2\alpha}_{y,\zeta} X_w f, f)_{L^2(\Sigma)}\right| \\
&= \sum_{|\alpha|\leq k} \left|(\chi X_w^* \partial^{2\alpha}_{y,\zeta} X_w f, f)_{L^2(\mathbb{R}^n)}\right| \\
&\leq \sum_{|\alpha|\leq k} \left|(\Lambda^{-|\alpha|+1/2}\chi X_w^* \partial^{2\alpha}_{y,\zeta} X_w f, \Lambda^{|\alpha|-1/2} f)_{L^2(\mathbb{R}^n)}\right| \\
&\leq C \|f\|^2_{H^{|\alpha|-1/2}(\mathbb{R}^n)}.
\end{aligned}$$

This proves the first part for $s \geq -1/2$ half-integer. The general case follows by interpolation. The other case is similar. □

Lemma 2.2.

(a) *Let $P \in \Psi(\mathbb{R}^n)$ properly supported and elliptic in $\overline{\Omega}$. Then for any s and $\ell < s$, $f \in H^s(\overline{\Omega})$.*

$$\|f\|_{H^s} \leq C_s \|Pf\|_{H^{s-m}} + C_{\ell,s}\|f\|_{H^\ell}.$$

(b) *The kernel of P on the space of the distributions with support in $\overline{\Omega}$ is finitely dimensional and consists of C_0^∞ functions.*

(c) *Assume in addition that P is injective on some closed subspace $\mathcal{L}$ of H^s. Then*

$$\|f\|_{H^s} \leq C_s \|Pf\|_{H^{s-m}}, \quad \forall f \in \mathcal{L}.$$

Proof.

(a) Using ellipticity $\exists Q \in \Psi^{-m}$ and $R \in \Psi^{-\infty}$ with $f = QPf + Rf$. Now, by Sobolev continuity $Q \colon H^{s-m} \to H^s$. So,

$$\|f\|_{H^s} \leq \|QPf\|_{H^s} + \|Rf\|_{H^s}$$

$$\leq \|Pf\|_{H^{s-m}} + \|Rf\|_{H^s}.$$

Since $\ell < s$, we have $H^s \subset H^\ell$. So, since R is a smoothing operator, we have $R\colon H^\ell \to H^s$. This implies the result.

(b) Let $f \in \ker P$. As before,

$$f = QPf + Rf = Rf.$$

Thus, $f \in \ker(\mathrm{Id} - R)$. By Fredholm alternative, $\dim\ker(I - R) < \infty$. By regularity we obtain the other result.

(c) Assume that the inequality is false. Then there exists a sequence $\|f_n\|_{H^s} = 1$ with $Pf_n \to 0$ in H^{s-m}. Since R is compact, there is a subsequence (also denoted by f_n) with Rf_n converging. Using (a), $f_n - f_m$ is Cauchy in H^s. Then, $f_n \to f \in H^s$. So, $Pf_n \to Pf = 0$ and $\|f\|_{H^s} = 1$, a contradiction.

□

This implies

Theorem 2.3. *Assume that $w \in C^\infty$ satisfies the ellipticity condition.*

a) $\exists C, C_s > 0$ such that for all $f \in L^2(\Omega)$

$$\|f\|_{L^2(\Omega)} \leq C \|X_w^* X_w f\|_{H^1(\Omega_1)} + C_s \|f\|_{H^{-s}(\mathbb{R}^n)}.$$

b) $\ker(X_w) \cap L^2(\Omega)$ is finitely dimensional and consists of $C_0^\infty(\mathbb{R}^n)$ functions.

c) If X_w is injective on $L^2(\Omega)$, then

$$\|f\|_{L^2(\Omega)} \leq C \|X_w^* X_w f\|_{H^1(\Omega_1)}.$$

Finally, we obtain the following result

Theorem 2.4 (Microlocal stability)**.** *Assume that $w \in C^\infty$ satisfies the ellipticity condition and that X_w is injective in L^2. Then, for any $f \in L^2(\Omega)$*

$$\|f\|_{L^2(\Omega)} \leq \|X_w f\|_{H^{1/2}(\Sigma)}.$$

2.2. **Visible and invisible singularities.**

Definition 2.5. Let $\Psi \in \Psi^m(\Omega)$ properly supported. We call $\mathrm{ES}(P)$ to be the set of *visible singularities* and the set where P has order $-\infty$ the set of *invisible singularities*.

We have the following generalization

Definition 2.6. Let $P\colon H^s_{\mathrm{comp}}(\Omega) \to H^{s-m}_{\mathrm{loc}}(\Omega)$ to be bounded. The set of *visible singularities* if the largest open conic set $\Gamma_1 \subset T^*\Omega \setminus 0$ such that

$$\chi(x, D)f = Qg + Rf,$$

holds, where $g = Pf$, $Q\colon H^s_{\mathrm{comp}}(\Omega) \to H^{s+m}_{\mathrm{loc}}(\Omega)$ and $\chi \in \Psi^0(\Omega)$ with $\mathrm{ES}(\chi) \subset \Gamma_1$. The *invisible singularities* is the largest open conic set $\Gamma_2 \subset T^*\Omega \setminus 0$ $\mathrm{WF}(f) \subset \Gamma_2$ implies $Pf \in C^\infty(\Omega)$.

Remark 2.7.

(1) On Γ_1 we have

$$\|\chi(x, D)f\|_{H^s} \leq C_s \|Pf\|_{H^{s-m}} + C_{l,s} \|f\|_{H^l},$$

for any $s \geq l$ and $f \in H^s_{\mathrm{comp}}$. So, the singularities on Γ_1 are not just visible, they are also *stably recoverable*.

(2) The previous inequality is false if $\mathrm{ES}(\chi) \subset \Gamma_2$, regardless of m, s and l.
(3) $\Gamma_1 \cup \Gamma_2$ may not cover $T^*\Omega \setminus 0$, even after closure.

Example 2.8. From Theorem 2.3, we see that the set of visible singularities of $X_w^* X_w$ is

$$\mathcal{V} = \{(x,\xi) \in T^*\mathbb{R}^n \setminus 0 : \exists \theta \perp \xi \text{ so that } w(x,\theta) \neq 0\},$$

where $\theta \perp \xi$ means $0 = \xi(\theta) = \xi_j \theta^j$. By Example 1.21, we see that $X_w^* X_w$ is smoothing away from $\mathcal{V}$. Hence, the set of invisible singularities is

$$\mathcal{U} = T^*\mathbb{R}^n \setminus \overline{\mathcal{V}},$$

where the closure is taken in the conic sense. So, to recover a singularity (x, ξ), we need to have a line trough x in a direction θ normal to ξ, so that $w(x,\theta) \neq 0$.

We also have the following corollary

Corollary 2.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\chi \in S^0$ with $\mathrm{ES}(\chi) \subset \mathcal{V}$.*

(a) For every s and $l < s$, we have for every $f \in H_0^s(\Omega)$

$$\|\chi(x,D)f\|_{H^s} \leq C_s \|X_w^* X_w\|_{H^{s+1}} + C_{s,l} \|f\|_{H^l}.$$

(b) The estimate on (a) does not hold when $\mathrm{ES}(\chi) \cap \mathcal{V} = \emptyset$, regardless of the choice of s and $l < s$.

Example 2.10. Assume that $X_w f(\ell)$ is known for lines ℓ in an open set $\mathcal{L}$. We can model such a case by choosing weights w constant along all lines (i.e., $\theta \cdot \nabla_x w = 0$), compactly supported in $\mathcal{L}$. In the Σ parameterization, this says that $w(x,\theta) = w_0(\theta, x - (\theta \cdot x)\theta)$ with some $w_0 \in C_0^\infty$. Then, the singularities that can be stably recovered are the ones in the conormal bundle of $\mathcal{L}$:

$$N^*\mathcal{L} = \{(x,\xi) \in N^*\ell : \ell \in \mathcal{L}\}.$$

Singularities outside $\overline{N^*\mathcal{L}}$ are not stably recoverable.

2.3. **Recovery on a region of interest.** Assume that we only want to recover f, given Xf or Rf, in a subdomain called a Region of Interest (ROI). If we know $Xf(\ell)$ for all lines through the ROI (or Rf for all planes through the ROI), does this determine f restricted to the ROI uniquely? The following example gives a negative answer.

Example 2.11. Given $a > 0$, let $g \in C_0^\infty(\mathbb{R})$ with $g(p) = 0$ for $|p| \leq a$. Then $g \in \mathcal{S}_H(\mathbb{R} \times S^1)$, i.e., satisfies

(i) $g(p,\omega) = g(-p,\omega)$;
(ii) $\mu_k g(\omega)$ is a homogeneous polynomial of degree k, for $k \in \mathbb{N}$.

Since $R\colon \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}_H(\mathbb{R} \times S^1)$ is a bijection, $\exists f \in \mathcal{S}(\mathbb{R}^2)$ with $g = Rf$. By the Support Theorem, f has compact support. Using the inversion formula obtained by expanding f and Rf on its Fourier series, we have

$$f(r) = -\frac{1}{\pi} \int_r^\infty (p^2 - r^2)^{-\frac{1}{2}} g'(p)\,dp.$$

To show that f is not identically zero, integrate by parts woth $r < a$ to obtain

$$f(r) = -\frac{1}{\pi} \int_r^\infty p(p^2 - r^2)^{-\frac{3}{2}} g(p)\,dp.$$

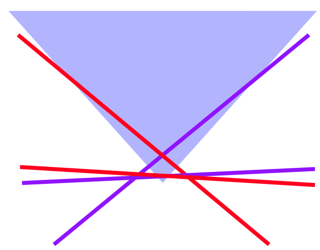
Now repeat with $g(p,\omega) = g_k(p) e^{ik \arg \omega}$, where $g_k \in C_0^\infty$ is even (odd) when k even (odd).

Example 2.12. Even though we cannot reconstruct f in the ROI uniquely, we can reconstruct it up to a smooth error. This allows us to recover the singularities. Indeed, let $\chi(\ell)$ be a smooth function on the line manifold equal to one on all lines through the ROI, and zero outside a larger neighborhood. Then $X_w^*\chi X_w f \in \Psi^{-1}$ is elliptic in the ROI if w is non-vanishing (see Example 1.21). Therefore, for $\psi \in C_0^\infty$ with $\chi = 1$ near the ROI and $\operatorname{supp}\psi$ close enough to the ROI, we have $\psi f = QX_w^*\chi X_w f + Rf$, where R is smoothing.

2.4. **Support theorems and injectivity.** We have the following non-uniqueness result for dimension 2.

Theorem 2.13 (Boman)**.**

(a) *There exists a smooth $w(x,\theta) > 0$ defined on $\mathbb{R}^2 \times S^1$ and $f \in C^\infty(\mathbb{R}^2)$ supported in $\{|x^1| \le x^2\}$ with $0 \in \operatorname{supp}$ so that $Xf(\ell) = 0$ where $\ell \in \{x^2 = ax^1 + b; |a| < 1, b < 1\}$.*



(b) *There exists a smooth $w(x,\theta) > 0$ defined on $\mathbb{R}^2 \times S^1$ and $f \in C^\infty(\mathbb{R}^2)$ not identically zero with $Xf(\ell) = 0$ for all lines.*

(c) *The set of smooth positive weights w for which (a) holds is dense in $C^k(\mathbb{R}^2 \times S^1)$ for every $k \in \mathbb{Z}_{\ge 0}$.*

In general, we have injectivity for small domains

Lemma 2.14. *Let $P \in \Psi^m(\Omega_1)$ be elliptic properly supported in a neighborhood Ω_1 of $\overline{\Omega}$. Then there exists $\varepsilon > 0$ so that for any open $U \subset \overline{\Omega}$ with measure $|U| < \varepsilon$, P is injective on distributions supported in U and*

$$\|f\|_{L^2(U)} \le C\,\|Pf\|_{H^{-m}(\Omega_1)} \quad \forall f \in L^2(U).$$

Proof.

- Let $Q \in \Psi^{-m}$ be a properly supported parametrix; then $QP = \mathrm{Id} + K$. Let $\mathcal{K}$ be the Schwartz kernel of K. Then,

$$\|Kf(x)\|_{L^2(U)} \le \left\| \int_U \mathcal{K}(x,y) f(y)\,dy \right\|_{L^2(U)} \le \|\mathcal{K}\|_{\mathrm{HS}}\,\|f\|_{L^2(U)}\,.$$

- Since $\mathcal{K}$ is uniformly bounded in $\overline{\Omega} \times \overline{\Omega}$, we have $\|\mathcal{K}\|_{\mathrm{HS}} \le C|U|$. Hence, for $|U| \ll 1$, $K\colon L^2(U) \to L^2(U)$ has a norm less than one. So, $I + K$ is invertible and $(I+K)^{-1}QP = I$ on $L^2(U)$.
- Any distribution $f \in \ker P$ would be C^∞ by ellipticity, therefore we get injectivity for distributions as well.

□

Theorem 2.15 (Injectivity on small domains)**.** *Let $w \in C^\infty(\overline{\Omega} \times S^{n-1})$ and assume that $w(x,\theta) \ne 0, \forall (x,\theta) \in \overline{\Omega} \times S^{n-1}$. Then there exists $\varepsilon > 0$ so that for any compact* *set $K \subset \overline{\Omega}$ with measure $|K| \le \varepsilon$, the weighted transform X_w is injective on $L^2(K)$ and*

$$\|f\|_{L^2} \le C \|X_w^* X_w f\|_{H^1(\Omega_1)}, \quad \forall f \in L^2(K).$$

Proof. Any smooth extension of w outside $\overline{\Omega}$ will not vanish for x close enough to $\partial\Omega$. Take one such extension and choose $\chi \in C_0^\infty(\Omega_1)$ equal to one in a neighborhood of the closure of the set where $w \neq 0$. Then $\chi X_w^* X_w \chi \in \Psi^{-1}(\Omega_1)$ is properly supported and elliptic, so we can apply the previous lemma. □

We still have a support theorem for $n \ge 3$, but first we need the following

Lemma 2.16. *Let X_w be injective on $L^2(\Omega)$, where with some nowhere vanishing $w \in C^\infty(\overline{\Omega}_1 \times S^{n-1})$ is nowhere vanishing. Then $\exists \varepsilon > 0$ so that $\forall v \in C^2(\overline{\Omega}_1 \times S^{n-1})$ with*

$$\|w - v\|_{C^2(\overline{\Omega}_1 \times S^{n-1})} \le \varepsilon,$$

then X_v is injective as well. Moreover, there is a constant $C = C(w) > 0$ with

$$\|f\|_{L^2(\Omega)} \le C \|X_v^* X_v f\|_{H^1(\Omega_1)},$$

for any such v.

Proof.

- We have $X_w^* X_w - X_v^* X_v = \mathrm{Op}(a)$, with

$$a(x, y, \theta) = \overline{w}(x, \theta) w(y, \theta) + \overline{w}(x, -\theta) w(y, -\theta) - \overline{v}(x, \theta) v(y, \theta) + \overline{v}(x, -\theta) v(y, -\theta).$$

  So, $\|a\|_{C^2(\overline{\Omega}_1 \times \overline{\Omega}_1 \times S^{n-1})} \le C \|w - v\|_{C^2(\overline{\Omega}_1 \times S^{n-1})}$, with $C = C(w)$
- Then $\|(X_w^* X_w - X_v^* X_v) f\|_{H^1(\Omega_1)} \le C\varepsilon \|f\|_{L^2(\Omega)}$. By Theorem 2.3

$$\|f\|_{L^2(\Omega)} \le C \|X_w^* X_w f\|_{H^1(\Omega_1)} \le \|X_v^* X_v f\|_{H^1(\Omega_1)} + C\varepsilon \|f\|_{L^2(\Omega)},$$

  and we can absorb the last term on the RHS by the one on the LHS for $\varepsilon \ll 1$.

□

Finally we have the following support theorem

Theorem 2.17. *Let $n \ge 3$, $w \in C^\infty(\mathbb{R}^n \times S^{n-1})$ be a nowhere vanishing. Let $f \in \mathcal{E}'(\mathbb{R}^n)$ with $X_w f(\ell) = 0$ for all lines ℓ not intersecting the ball $B(0, R)$. Then $f(x) = 0$ for $|x| > R$. In particular, X_w is injective.*

Proof. Let $\rho = \max_{\operatorname{supp} f} |x|$. WLOG, $\theta = e_n$. Let $\pi_{s,\theta} = \{x | x \cdot \theta = s\}$, $w_s(x', \theta') := w((x', s), (\theta', 0))$.

- For δ small enough, $X_{w_s}^* X_{w_s} : L^2(B^{n-1}(0, \delta)) \to B^{n-1}(0, 1)$ is injective, for $s = \rho$.
- Since injectivity is conserved under small perturbations, the same is true for s close to ρ.
- Then, $f = 0$ on $\pi_{s,\theta} \cap B(0, \rho)$, s close to ρ.
- $f = 0$ in a neighborhood of $\rho\theta$. Now repeat and take a finite cover of S^{n-1}.

□

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