Milnor-Friedlander's problem for diffeomorphisms

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Inspiring work of Milnor on Lie groups

Let G be a Lie group and let G^δ denote the same group with the discrete topology. The natural homomorphism from G^δ to G induces a continuous mapping $\eta : BG^\delta \to BG$.

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Conjecture (Isomorphism conjecture)

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- Friedlander posed a similar conjecture for Lie groups defined over algebraically closed fields.
- Milnor proved the isomorphism conjecture for solvable Lie groups.
- We can ask a similar question for other topological group $G = \mathrm{Diff}(M), Homeo(M), Symp(M,\omega), Ham(M,\omega), \ldots$.

Theorem (Milnor)

For any Lie group with finitely many connected components, the induced maps

$$H^*(BG; \mathbb{F}_p) \to H^*(BG^\delta; \mathbb{F}_p),$$

$$H^*(BG; \mathbb{Z}) \to H^*(BG^\delta; \mathbb{Z}),$$

are injective.

Idea of the proof.

Becker-Gottlieb transfer for the map $BN \to BG$ where N is the normalizer of the maximal torus. □

Stable result for Lie groups

Suslin proved the isomorphism conjecture for $GL_n(\mathbb{C})$ where , in the stable range:

Theorem (Suslin)

The natural map

$$BGL_n(\mathbb{C})^\delta \to BGL_n(\mathbb{C})$$

induces isomorphisms

$$H_i(BGL_n(\mathbb{C})^\delta; \mathbb{F}_p) \xrightarrow{\simeq} H_i(BGL_n(\mathbb{C}); \mathbb{F}_p).$$

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Question

Does the map $\eta : \mathrm{BDiff}^\delta(M) \to \mathrm{BDiff}(M)$ induce an injective map in cohomology? Is there a "range" depending on M that η induces an injective map on cohomology or surjective map in homology?

What is so different about $G = \mathrm{Diff}(M)$?

For a manifold M, the group of diffeomorphisms $\mathrm{Diff}^{\delta}(M)$ contains "the information" of two very different groups

$$1 \to \mathrm{Diff}_0(M) \to \mathrm{Diff}(M) \to \mathrm{MCG}(M) \to 1$$

- The group $\mathrm{Diff}_0(M)$ is an interesting object from dynamical system and foliation point of view.
- The group $\mathrm{MCG}(M)$ is an interesting object from geometric topology point of view.

$M = S^1$

$$\eta : \mathrm{BDiff}^\delta(S^1) \to \mathrm{BDiff}(S^1) \simeq \mathbb{C}P^\infty$$

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Theorem (Morita, '85)

For all $k \geq 1$, there is a surjective map

$$H_{2k}(\mathrm{Diff}^{\delta}(S^1); \mathbb{Z}) \to \mathbb{Z} \oplus S^k_{\mathbb{Q}}\mathbb{R}.$$

A lost theorem of Thurston

Let $\mathrm{Diff}^{\omega}(S^1)$ denote the analytic diffeomorphisms of the circle. Thurston claimed that for all flat analytic S^1-bundle on 6-manifolds the cube of the Euler class is zero. This means that the map

$$H^6(\mathbb{C}P^{\infty};\mathbb{Q}) \to H^6(\mathrm{BDiff}^{\omega,\delta}(S^1);\mathbb{Q})$$

is zero. However, it is not hard to show that

$$H^6(\mathbb{C}P^{\infty};\mathbb{Z}) \to H^6(\mathrm{BDiff}^{\omega,\delta}(S^1);\mathbb{Z})$$

is not zero!

M = surface

Let $\Sigma_{g,k}$ denote a surface of genus g and k boundary components. We consider the following cases:

$$B\mathrm{Diff}^{\delta}(\Sigma_{g,k}, \partial) \to B\mathrm{Diff}(\Sigma_{g,k}, \partial)$$

$$B\mathrm{Symp}^{\delta}(\Sigma_{g,k}, \partial) \to B\mathrm{Symp}(\Sigma_{g,k}, \partial)$$

$$B\mathrm{Diff}^{\delta}(\mathbb{D}^2 - n \text{ points}, \partial) \to B\mathrm{Diff}(\mathbb{D}^2 - n \text{ points}, \partial)$$

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Remark

$$B\mathrm{Diff}(\Sigma_{g,k}, \partial) \simeq B\mathrm{MCG}(\Sigma_{g,k}) \text{ Earle-Eells Theorem}$$

$$B\mathrm{Diff}(\Sigma_{g,k}, \partial) \simeq B\mathrm{Symp}(\Sigma_{g,k}, \partial) \text{ Moser's Theorem}$$

$$B\mathrm{Diff}(\mathbb{D}^2 - n \text{ points}, \partial) \simeq B\mathrm{Br}_n$$

.

Main theorems

Theorem (N)

The induced maps on cohomology

$$H^*(\mathrm{BDiff}(\Sigma_{g,k},\partial);\mathbb{F}_p) \to H^*(\mathrm{BDiff}^\delta(\Sigma_{g,k},\partial);\mathbb{F}_p)$$

$$H^*(\mathrm{BSymp}(\Sigma_{g,k},\partial);\mathbb{F}_p) \to H^*(\mathrm{BSymp}^\delta(\Sigma_{g,k},\partial);\mathbb{F}_p)$$

are injective for $* \leq (2g-2)/3$.

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Theorem (N)

In the case of punctured disk, we have

$$H^*(B\mathrm{Diff}(\mathbb{D}^2 - n\ points, \partial);\mathbb{Z}) \to H^*(B\mathrm{Diff}^\delta(\mathbb{D}^2 - n\ points, \partial);\mathbb{Z})$$

is injective in all degrees.

The idea: $\coprod_n B\mathrm{Br}_n$ is a free E_2-algebra!

Idea of the proof

Step 1: We know that $B\mathrm{Diff}(\Sigma_{g,k}, \partial)$ exhibits homological stability (Harer). So we showed that $B\mathrm{Diff}^{\delta}(\Sigma_{g,k}, \partial)$ is also homologically stable (Morita's problem).

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Step 1: We know that $B\mathrm{Diff}(\Sigma_{g,k}, \partial)$ exhibits homological stability (Harer). So we showed that $B\mathrm{Diff}^{\delta}(\Sigma_{g,k}, \partial)$ is also homologically stable (Morita's problem).
Step 2: Let $G = \mathrm{Diff}(\Sigma_{\infty,k}, \partial)$,

$$\begin{array}{ccc} BG^{\delta} & \xrightarrow{\quad\eta\quad} & BG \\ \downarrow & & \downarrow \\ ? & \longrightarrow & \Omega_0^{\infty}\mathrm{MTSO}(2) \end{array}$$

where the righthand vertical map is a homology isomorphism (Madsen-Weiss Theorem).
Recall MTSO(2) is the Thom spectrum of of the virtual bundle $-\gamma$, where γ is the tautological bundle over $BGL_2^{+}(\mathbb{R})$.

Haefliger spaces

Definition

Let Γ_2 denote the topological groupoid whose **objects are** $\mathbb{R}^2$ and whose **morphisms are germs** of orientation preserving diffeomorphisms (with sheaf topology). The classifying space of this groupoid is the Haefliger space of oriented codimension two foliations.

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Definition

Let $\mathrm{MT}\nu$ be the Thom spectrum of the virtual bundle $\nu^*(-\gamma)$.

We showed that there is a map

$$BG^\delta \to \Omega_0^\infty \mathrm{MT}\nu$$

which induces a homology isomorphism.

Idea of the proof continued

Step 3: One can use maps of groupoids

$$S^{1^\delta} \to \Gamma_2 \to GL_2^+(\mathbb{R}),$$

to show that the natural map

$$\Omega_0^\infty \mathrm{MT}\nu \to \Omega_0^\infty \mathrm{MTSO}(2)$$

has a section after p-completion.

MMM-classes

One can use the main theorem to show that the map

$$\mathbb{Z}[\kappa_1, \kappa_2, \dots] \to H^*(BG^\delta; \mathbb{Z})$$

is injective. However, Morita showed that κ_i in $H^*(BG^\delta; \mathbb{Q})$ is zero for $i > 2$. This implies that the natural map

$$H^*(BG^\delta; \mathbb{Z}) \otimes \mathbb{Q} \to H^*(BG^\delta; \mathbb{Q})$$

has a huge kernel. Also for $i > 2$ one can use Cheeger-Simons theory to define MMM-characters

$$\hat{\kappa}_i : H_{2i-1}(BG^\delta; \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$$

which maps to κ_i via the Bockstein map.

Other characteristic classes

- Secondary characteristic classes: One can use these methods to show that

$$H_3(\mathrm{BDiff}^{\delta}(D^2,\partial);\mathbb{Q}) \to H_3(BG^{\delta};\mathbb{Q})$$

  is surjective.
- Morita and Kotschick showed that there is a surjective map

$$H_{2k}(\mathrm{Symp}^{\delta}(\Sigma,\partial);\mathbb{Q}) \to \mathbb{Q} \oplus S^2\mathbb{R} \oplus \cdots \oplus S^k(S^2\mathbb{R}).$$

  for $g(\Sigma) \geq 3k+1$.

- We can give a more direct proof of the Morita-Kotschick theorem by using the universal space $B\Gamma_2^{\text{vol}}$. We used these methods to show that

$$H_2(\mathrm{Symp}^{\delta}(\Sigma,\partial);\mathbb{Z}[\frac{1}{6}]) \xrightarrow{\simeq} H_4(B\Gamma_2^{\text{vol}};\mathbb{Z}[\frac{1}{6}]),$$

and it is not hard to see that $H_4(B\Gamma_2^{\text{vol}};\mathbb{Z})$ surjects to $\mathbb{Z}\oplus S^2_{\mathbb{Q}}\mathbb{R}$.

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  and it is not hard to see that $H_4(B\Gamma_2^{\mathrm{vol}}; \mathbb{Z})$ surjects to $\mathbb{Z} \oplus S^2_{\mathbb{Q}}\mathbb{R}$.
- There is a short exact sequence

$$1 \to \widetilde{\mathrm{Ham}}^\delta(\Sigma, \partial) \to \mathrm{Symp}^\delta(\Sigma, \partial) \to H^1(\Sigma; \mathbb{R}) \to 1.$$

  Bowden observed that all $\kappa_i \in H^*(\widetilde{\mathrm{Ham}}^\delta(\Sigma, \partial); \mathbb{Q})$ are zero.

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On a manifold M all plane fields of dim $\leq \lfloor (\dim(M) + 1)/2 \rfloor$ are integrable up to homotopy.

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Conjecture

The map

$$H_*(\mathrm{BDiff}^{\delta}(M);\mathbb{Z}) \to H_*(\mathrm{BDiff}(M);\mathbb{Z})$$

is an isomorphism for $* \leq \dim(M)$.

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is an isomorphism for $* \leq \dim(M)$.

- For $M = \mathbb{R}^n$, it is implied by a theorem of Segal.
- Thurston proved the case $* = 1$.
- Thurston proved the bound is optimum for C^2 diffeomorphisms of $M = S^n$.
- Thurston proved that the map

$$BHomeo^\delta(M) \to BHomeo(M)$$

induces a homology isomorphism in all degrees!

Happy birthday dear Michael! ☺